

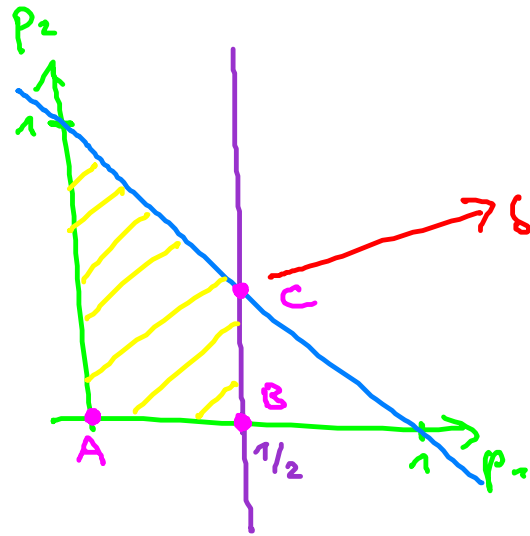
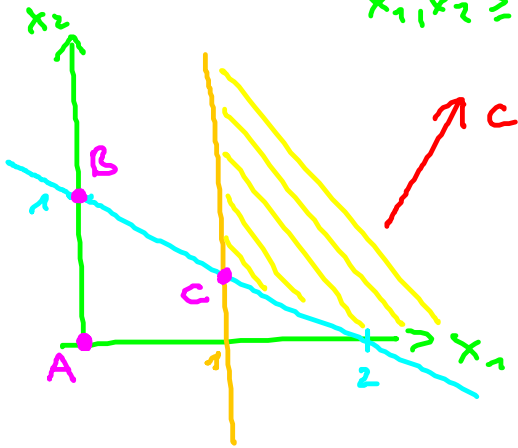
Example:

$$\begin{aligned} \min \quad & x_1 + x_2 \\ \text{s.t.} \quad & x_1 + 2x_2 - x_3 = 2 \\ & x_1 - x_4 = 1 \\ & x_1, \dots, x_4 \geq 0 \end{aligned}$$

$$\begin{aligned} \max \quad & 2p_1 + p_2 \\ \text{s.t.} \quad & p_1 + p_2 \leq 1 \\ & 2p_1 \leq 1 \\ & p_1, p_2 \geq 0 \end{aligned}$$

↕ equivalent

$$\begin{aligned} \min \quad & x_1 + x_2 \\ \text{s.t.} \quad & x_1 + 2x_2 \geq 2 \\ & x_1 \geq 1 \\ & x_1, x_2 \geq 0 \end{aligned}$$



$$\begin{array}{c|cccc} \text{A} & 0 & 1 & 1 & 0 & 0 \\ \hline x_3 & -2 & -1 & -2 & 1 & 0 \\ x_4 & -1 & -1 & 0 & 0 & 1 \end{array}$$

$$\begin{array}{c|cccc} \text{B} & -1 & 1/2 & 0 & 1/2 & 0 \\ \hline x_2 & 1 & 1/2 & 1 & -1/2 & 0 \\ x_4 & -1 & -1 & 0 & 0 & 1 \end{array}$$

$$\begin{array}{c|cccc} \text{C} & -3/2 & 0 & 0 & 1/2 & 1/2 \\ \hline x_2 & 1/2 & 0 & 1 & -1/2 & 1/2 \\ x_1 & 1 & 1 & 0 & 0 & -1 \end{array}$$

Conclusions: Consider primal LP in standard form and its dual:

(i) Every basis B determines a basic solution to the primal ($x_B = B^{-1} \cdot b$, $x_i = 0$ for $i \notin B(1), \dots, B(m)$) but also to the dual: $p^T = c_B^T \cdot B^{-1}$.

(ii) p is feasible $\Leftrightarrow \bar{c} \geq 0$

(iii) Reduced cost $\bar{c}_i = 0$ corresponds to active dual constraint.

(iv) p is degenerate $\Leftrightarrow \bar{c}_i = 0$ for some non-basic variable x_i

4.6 Farkas' lemma

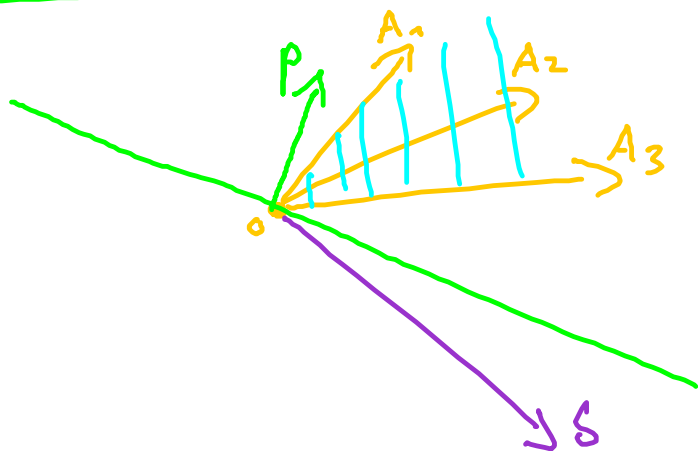
Theorem (Farkas' lemma)

For $A \in \mathbb{R}^{m \times n}$ and $b \in \mathbb{R}^m$ exactly one of the following two alternatives holds:

(i) $\exists x \geq 0 : Ax = b$

(ii) $\exists p : p^T A \geq 0$ and $p^T b < 0$

Geometric illustration:



Proof: (i) \Rightarrow \neg (ii):

Let $p \in \mathbb{R}^m$ with $p^T A \geq 0$. Then

$$p^T \cdot b = \underbrace{(p^T \cdot A)}_{\geq 0} \underbrace{x}_{\geq 0} \geq 0$$

\neg (i) \Rightarrow (ii):

$$\begin{aligned} \max \quad & c^T \cdot x \\ \text{s.t.} \quad & Ax = b \\ & x \geq 0 \end{aligned}$$

infeasible. Dual:

$$\begin{aligned} \min \quad & p^T b \\ \text{s.t.} \quad & p^T A \geq 0 \end{aligned} \quad \begin{array}{l} \text{unbounded or} \\ \text{infeasible} \end{array}$$

$p=0$ feasible solution to dual LP

\Rightarrow dual is unbounded.

$\Rightarrow \exists p : p^T A \geq 0$ and $p^T b < 0$. \square

Theorem: Let $Ax \leq b$ feasible and $d \in \mathbb{R}$.
 $c \in \mathbb{R}^n$

The following are equivalent:

(i) $\forall x : Ax \leq b \Rightarrow c^T x \leq d$

(ii) $\exists p \geq 0 : p^T A = c^T$ and $p^T \cdot b \leq d$

Proof: Consider primal-dual pair

$\max c^T x$

$\min p^T \cdot b$

s.t. $Ax \leq b$

s.t. $p^T A = c^T$

$p \geq 0$

(i) \Leftrightarrow opt cost for primal LP $\leq d$

(ii) \Leftrightarrow " " " dual " $\leq d$

\square

4.8 Cones and extreme rays

Def: A set $C \subseteq \mathbb{R}^n$ is a cone if $\lambda \cdot x \in C$ for all $x \in C$ and $\lambda \geq 0$.

Notice that $0 \in C$ for every nonempty cone C .

Example: A polyhedron of the form

$$P = \{x \mid Ax \geq 0\}$$

is a polyhedral cone.

Notice that $x \in P$ with $x \neq 0$ is not a vertex of P :

$$x = \frac{1}{2} \cdot \left(\underbrace{\frac{1}{2}x}_{\in P} + \underbrace{\frac{3}{2}x}_{\in P} \right) \quad (\text{convex combination})$$

$\Rightarrow 0$ is the only possible vertex of a polyhedral cone.

If $0 \in P$ is a vertex, then P is pointed.

Theorem: Let $C = \{x \mid a_i^T x \geq 0, i=1, \dots, m\}$. The following are equivalent:

- (i) 0 is an extreme point of C
- (ii) C does not contain a line
- (iii) There exist n linearly independent vectors in $\{a_i \mid i=1, \dots, m\}$.

Proof: See Sect. 2.5. \square

