4.8 Cones and extreme rays

$C \subseteq \mathbb{R}^n$ cone if $\forall x \in C, \lambda \geq 0 : \lambda \cdot x \in C$

$\Rightarrow 0 \in C$ if $C \neq \emptyset$

$C = \{x \in \mathbb{R}^n \mid A \cdot x \geq 0\}$ polyhedral cone

0 is the only possible vertex of a polyhedral cone.

If 0 is vertex of C, then C is pointed.

Theorem: Equivalent:

(i) 0 is extreme point of C

(ii) C does not contain a line

(iii) A has n linearly indep. rows.

Let $P = \{x \in \mathbb{R}^n \mid Ax \geq b\}$ and $y \in P$. The recession cone (at y) of P is $\{d \in \mathbb{R}^n \mid \underbrace{A \cdot (y + \lambda \cdot d) \geq b}_{y + \lambda \cdot d \in P} \ \forall \lambda \geq 0\}$

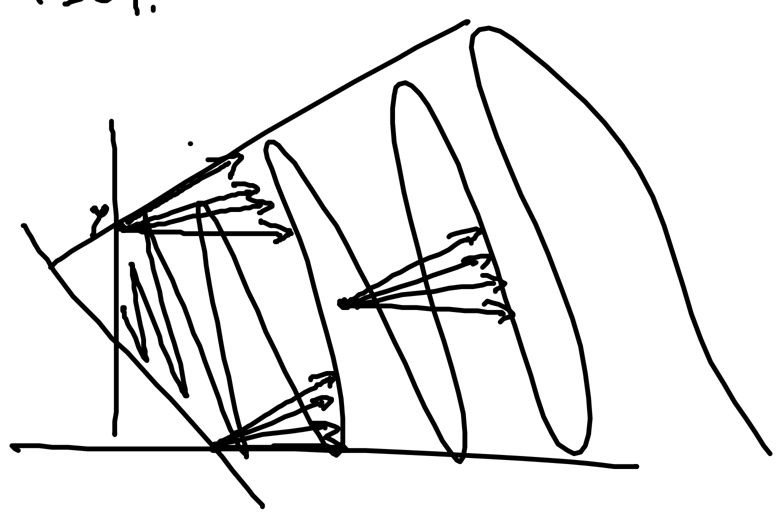
$= \{d \in \mathbb{R}^n \mid A \cdot d \geq 0\}$ polyhedral cone.

The nonzero elements of the recession cone are called the rays of P.

The recession cone of $P = \{x \in \mathbb{R}^n \mid Ax = b, x \geq 0\}$ is $\{d \in \mathbb{R}^n \mid A \cdot d = 0, d \geq 0\}$.

$A \cdot (y + \lambda \cdot d)$

$= \underbrace{A \cdot y}_{= b} + \lambda \cdot Ad$



Extreme rays

Def: (i) A nonzero element x of a polyhedral cone C is an <u>extreme ray</u>, if there are $n-1$ linearly indep. constraints that are active at x.

(ii) An extreme ray of the recession cone of some polyhedron P is also called an extreme ray of P.

Remark: Up to multiplication with positive vectors there are only finitely many extreme rays of a polyhedron.

Characterization of unbounded linear programs

Theorem: Let $C := \{x \in \mathbb{R}^n \mid a_i^T x \geq 0, i = 1, \dots, m\}$ a pointed polyhedral cone and $c \in \mathbb{R}^n$. The minimal cost $c^T x$ s.t. $x \in C$ is equal to $-\infty \iff$ There exists an extreme ray d of C with $c^T \cdot d < 0$.

Proof: „$\Leftarrow$" trivial ✓

„$\Rightarrow$": $\quad \text{opt} = -\infty \Rightarrow \exists x \in C : c^T x < 0$

$$\Rightarrow \exists x \in C : c^T x = -1$$

$$\Rightarrow P := \{x \in \mathbb{R}^n \mid a_i^T x \geq 0, i = 1, \dots, m, \; c^T x = -1\} \neq \emptyset$$

P has at least one extreme point since there are n linearly independent rows in the linear system defining the polyhedron. Let $d \in P$ be an extreme point. $\Rightarrow$ There are n linearly independent active constraints. $\Rightarrow$ Among the constraints $a_i^T x \geq 0$ $i = 1, \dots, m$ there are $n-1$ that are active.

$\Rightarrow d$ is extreme ray $\blacksquare$

Theorem: The last theorem also holds if C is a polyhedron with at least one extreme point.

Proof: Let $C = \{x \in \mathbb{R}^n \mid Ax \geq b\}$

"$\Leftarrow$" clear.

"$\Rightarrow$": Consider infeasible dual LP:

$$\begin{aligned} \max\ & p^T \cdot b \\ \text{s.t.}\ & p^T A = c^T \\ & p \geq 0 \end{aligned}$$

Changing objective function, the LP remains infeasible:

$$\begin{aligned} \max\ & p^T \cdot 0 \\ \text{s.t.}\ & p^T A = c^T \\ & p \geq 0 \end{aligned}$$

$\longrightarrow$ associated primal LP is infeasible or unbounded.

$$\begin{aligned} \min\ & c^T x \\ \text{s.t.}\ & Ax \geq 0 \end{aligned}$$

not infeasible since $x = 0$ is feasible solution

$\Rightarrow$ unbounded.

$\Rightarrow$ By the last theorem, there exist an extreme ray d of $\{x \mid Ax \geq 0\}$ with $c^T d < 0$.

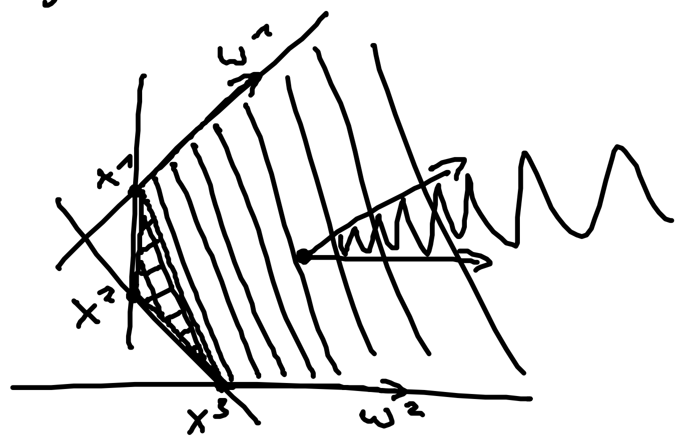
d is by def. also an extreme ray of P. $\square$

Remark: If the simplex method observes that an LP is unbounded, the corresponding j-th basic direction is an extreme ray d with $c^T d < 0$.

4.9 Representation of polyhedra

Theorem: Let $P = \{x \in \mathbb{R}^n \mid Ax \geq b\} \neq \emptyset$ with at least one extreme point. Let $x^1, \dots, x^k$ be the extreme points of P and $w^1, \dots, w^r$ a complete set of extreme rays of P. Then:

$$P \overset{!}{=} \{ \sum_{i=1}^{k} \lambda_i x^i + \sum_{j=1}^{r} \theta_j w^j \mid \lambda_i, \theta_j \geq 0, \sum_{i=1}^{k} \lambda_i = 1 \} =: Q$$



Proof: „$\supseteq$" clear.

„$\subseteq$": Assume by contradiction that $z \in P \setminus Q$. Consider following LP:

$$\begin{aligned} \max\ & 0 \\ \text{s.t.}\ & \sum_{i=1}^{k} \lambda_i x^i + \sum_{j=1}^{r} \theta_j w^j = z \\ & \sum_{i=1}^{k} \lambda_i = 1 \\ & \lambda, \theta \geq 0 \end{aligned}$$

Infeasible since $z \notin Q$. Consider dual LP

$$\begin{aligned} \min\ & p^T \cdot z + q \cdot 1 \\ \text{s.t.}\ & p^T \cdot x^i + q \geq 0 \\ & p^T \cdot w^j \geq 0 \end{aligned}$$

feasible ($p = 0, q = 0$)

$\Rightarrow$ unbounded $\Rightarrow \exists$ feasible solution (p, q) with $p^T \cdot z + q < 0$ $\Longleftrightarrow$ $p^T \cdot x^i + q \geq 0$

$p^T \cdot w^j \geq 0$

Consider LP:

$$\min \; p^T \cdot x$$
$$\text{s.t. } Ax \geq b$$

z is a feasible sol. (since $z \in P$)

If LP is bounded $\Rightarrow$ opt. sol. x^i for some i ↯

$p^T z < p^T \cdot x^i \;\; \forall i$

$\Rightarrow$ LP is unbounded $\Rightarrow \exists$ extreme ray w^j with $p^T \cdot w^j < 0$ ↯.

$\square$

<u>Corollary:</u> A nonempty bounded polyhedron is the convex hull of its extreme points.

<u>Corollary:</u> If $C = \{x \mid Ax \geq 0\}$ pointed, then every $x \in C$ is a nonnegative linear combination of extreme rays of C.

<u>Theorem:</u> (without proof — see book)

For $x^1, \dots, x^k \in \mathbb{R}^n$, $w^1, \dots, w^r \in \mathbb{R}^n$ the set

$$Q := \left\{ \sum_{i=1}^{k} \lambda_i x^i + \sum_{j=1}^{r} \theta_j \cdot w^j \;\middle|\; \lambda_i, \theta_j \geq 0, \; \sum_{i=1}^{k} \lambda_i = 1 \right\}$$

is a polyhedron.