<u>Conclusion</u> There are two ways of representing a polyhedron

(i) In terms of a finite set of linear constraints, e.g.

$$P = \{ x \in \mathbb{R}^n \mid Ax \geq b \}$$

"outer representation"

(ii) As a <u>finitely generated set</u> in terms of its extreme points and extreme rays, i.e.

$x^1, \dots, x^k, \; w^1, \dots, w^r$

$$P = \left\{ \sum_{i=1}^{k} \lambda_i x^i + \sum_{j=1}^{r} \Theta_j \cdot w^j \;\middle|\; \lambda_i, \Theta_j \geq 0, \sum_{i=1}^{k} \lambda_i = 1 \right\}$$

"inner representation"

Chapter 5: Sensitivity Analysis

5.1 Local sensitivity analysis

$$\begin{array}{llll} \min & c^T x & \max & p^T b \\ \text{s.t.} & Ax = b & \text{s.t.} & p^T A \leq c^T \\ & x \geq 0 & & \end{array}$$

Let B be an optimal basis, i.e.

$B^{-1} \cdot b \geq 0$ (feasibility)

$c^T - c_B^T \cdot B^{-1} \cdot A \geq 0$ (optimality) with corresponding opt. basic solution x^*.

Questions:

(i) Under what conditions does B remain optimal when part of the problem data is being changed?

(ii) What if B is no longer optimal after that change?

A new variable is added

$$\longrightarrow \quad \min \; c^T x + c_{n+1} \cdot x_{n+1}$$
$$\text{s.t.} \quad A \cdot x + A_{n+1} \cdot x_{n+1} = b$$
$$x, x_{n+1} \geq 0$$

$(x^*, 0)$ is a basic feasible solution to the new problem. B remains optimal if $\bar{c}_{n+1} = c_{n+1} - c_B^T \cdot B^{-1} \cdot A_{n+1} \geq 0$.

Otherwise, apply primal simplex method to reoptimize.

A new inequality constraint is added

$$a_{m+1}^T \cdot x \geq b_{m+1}$$

If x^* satisfies this inequality then x^* is still optimal for the new LP.

Otherwise, introduce slack variable $x_{n+1} \geq 0$ and rewrite

$$a_{m+1}^T \cdot x - x_{n+1} = b_{m+1}$$

$\rightarrow$ matrix A is replaced by $\bar{A} = \begin{pmatrix} A & 0 \\ a_{m+1}^T & -1 \end{pmatrix}$

$\rightarrow$ new basis $\bar{B} = \begin{pmatrix} B & 0 \\ a^T & -1 \end{pmatrix}$ with associated basic solution $(x^*, \underbrace{a_{m+1}^T \cdot x^* - b_{m+1}}_{<0})$

$$\bar{B}^{-1} = \begin{pmatrix} B^{-1} & 0 \\ a^T B^{-1} & -1 \end{pmatrix}$$

new reduced cost vector:

$$(c^T \; 0) - (c_B^T \; 0) \cdot \bar{B}^{-1} \cdot \bar{A} = (c^T - c_B^T \cdot B^{-1} \cdot A \, , \, 0) \geq 0$$

$\rightarrow$ apply dual simplex method to reoptimize.

<u>A new linear equation is added</u>

$$a_{m+1}^T \cdot x = b_{m+1}$$

If x^* satisfies this equation, then it is still optimal.

Otherwise, w.l.o.g. $a_{m+1}^T x^* > b_{m+1}$

Consider auxiliary problem

$$\begin{aligned} \min \quad & c^T x + M \cdot x_{n+1} \qquad M > 0 \text{ large} \\ \text{s.t.} \quad & Ax = b \\ & a_{m+1}^T \cdot x - x_{n+1} = b_{m+1} \\ & x, x_{n+1} \geq 0 \end{aligned}$$

→ new basis $\bar{B}$ by adding new row and new column corresponding to x_{n+1}.
$\bar{B}$ is feasible basis.
→ use primal simplex method to re-optimize.

<u>Change the right-hand side b</u>

b is changed to $b + \gamma \cdot e_i$ (only b_i changes)

The optimality condition $\bar{c} \geq 0$ is not affected by this change.

Feasibility: $B^{-1} \cdot (b + \gamma \cdot e_i) \geq 0$?

Let $g = (\beta_{1i}, \dots, \beta_{mi})$ be the i-th column of B^{-1}

$$B^{-1} \cdot (b + \gamma \cdot e_i) = x_B^* + \gamma \cdot g \geq 0$$

$$\Leftrightarrow x_{B(j)}^* + \gamma \cdot \beta_{ji} \geq 0 \text{ for } j = 1, \dots, m$$

$$\Leftrightarrow \max_{j : \beta_{ji} > 0} - \frac{x_{B(j)}^*}{\beta_{ji}} \leq \gamma \leq \min_{j : \beta_{ji} < 0} - \frac{x_{B(j)}^*}{\beta_{ji}}$$

Otherwise, use dual simplex method to reoptimize.

<u>Changes in the cost vector</u>

c_j is changed to $c_j + \gamma$

feasibility is not affected but optimality

1. Case: x_j is non-basic -> only $\bar{c}_j$ is affected. New reduced cost of x_j:

$$\hat{c}_j = c_j + \gamma - c_B^T \cdot B^{-1} \cdot A_j = \bar{c}_j + \gamma$$

=> B remains optimal <=> $\gamma \geq -\bar{c}_j$

2. Case: $\underset{}{x_j}^{=x_{B(\ell)}}$ is basic -> all reduced cost coefficients are affected

c_B becomes $c_B + \gamma \cdot e_\ell$

$$(c_B^T + \gamma \cdot e_\ell^T) \cdot B^{-1} \cdot A_i \leq c_i \quad \forall i \neq j \ ?$$

$$\Longleftrightarrow \gamma \cdot q_{\ell i} \leq \bar{c}_i \quad \forall i \neq j$$

where $q_{\ell i}$ is ℓ-th entry of $B^{-1} \cdot A_i$

Otherwise, reoptimize with primal simplex method.

Changes in nonbasic column of A

Entry a_{ij} of A_j is changed to $a_{ij} + \gamma$

B is still feasible but $\bar{c}_j$ is affected

$$c_j - \underbrace{p^T}_{c_B^T \cdot B^{-1}} \cdot (A_j + \gamma \cdot e_i) = \bar{c}_j - \gamma \cdot p_i \geq 0 \ ?$$

Otherwise reoptimize using primal simplex method.

Changes in a basic column of A

see exercises.

5.2 Global dependence on the right-hand side

$$P(b) = \{x \in \mathbb{R}^n \mid Ax = b,\ x \geq 0\}$$

$$S = \{b \mid P(b) \neq \emptyset\} = \{Ax \mid x \geq 0\}$$

$$F(b) := \min_{x \in P(b)} c^T x \quad \text{for } b \in S$$

Assume that dual feasible set

$$\{p \mid p^T A \leq c^T\} \neq \emptyset \implies F(b) > -\infty \ \forall b \in S$$

S is convex. Let $b^1, b^2 \in S$ and $\lambda \in [0,1]$

$$\lambda \cdot b^1 + (1-\lambda) \cdot b^2 \in S \ ?$$

$$\exists x^1, x^2 \geq 0 : b^1 = Ax^1,\ b^2 = Ax^2$$

$$\lambda \cdot b^1 + (1-\lambda) \cdot b^2 = \lambda A x^1 + (1-\lambda) \cdot A x^2$$

$$= A \cdot \underbrace{(\lambda x^1 + (1-\lambda) x^2)}_{\geq 0} \in S$$

Consider fixed $b^* \in S$ and assume that B is a non-degenerate optimal basis

$x_B = B^{-1} \cdot b^* > 0$

$\bar{c}^T = c^T - c_B^T \cdot B^{-1} \cdot A \geq 0.$

Changing b^* to b with $b - b^*$ sufficiently small, $B^{-1} \cdot b$ remains positive and B is still optimal.

$\Rightarrow F(b) = c_B^T \cdot B^{-1} \cdot b = p^T \cdot b$ for b close to b^*

$\longrightarrow$ F is linear in the vicinity of b^*, its gradient is equal to the dual solution p.