5.2 Global dependence on the right-hand side

$$\begin{aligned} \min\ & c^T x \\ \text{s.t. } & Ax = b \\ & x \geq 0 \end{aligned} \qquad P(b) = \{x \mid Ax = b, x \geq 0\}$$

$$S = \{b \mid P(b) \neq \emptyset\} \quad \text{convex}$$

$$F(b) := \min_{x \in P(b)} c^T x \quad \forall b \in S$$

Assume that dual feasible set is nonempty

$$\{p \mid p^T A \leq c^T\} \neq \emptyset$$

$\Rightarrow F(b) > -\infty \quad \forall b \in S$

$b^* \in S$ with non-degenerate optimal Basis B

Then F is linear in the vicinity of b^* and the gradient of F in b^* is equal to the dual solution p.

Theorem: $F(b)$ is a convex and piecewise linear function of b.

Proof: Consider the feasible dual LP

$$\begin{aligned} \max\ & p^T b \\ \text{s.t. } & p^T A \leq c^T \end{aligned} \qquad \begin{aligned} & \operatorname{rank}(A) = m \\ & \Rightarrow \text{extreme points do exist.} \end{aligned}$$

Let $p^1, \dots, p^N$ denote the extreme points of $\{p \mid p^T A \leq c^T\}$

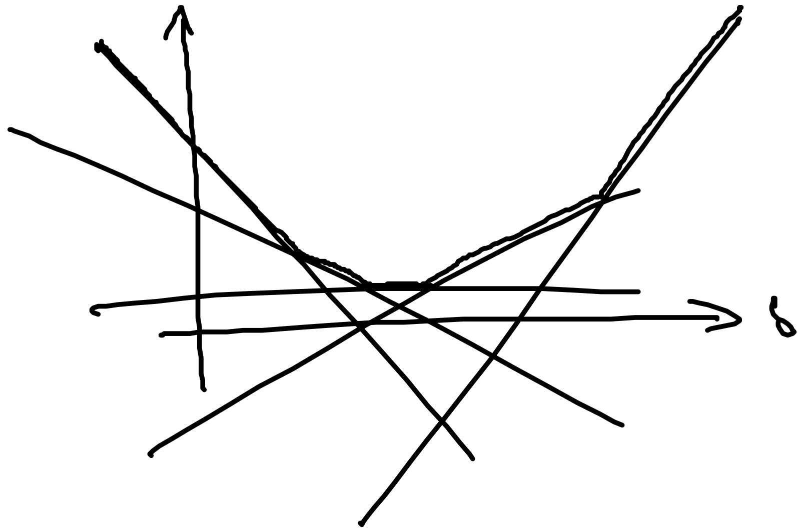
By strong duality

$$F(b) = \max_{i=1,\dots,N} (p^i)^T \cdot b \qquad \forall b \in S$$

$\Rightarrow F$ is maximum of convex (linear) functions

$\Rightarrow F$ is convex. $\square$

$F(b)$

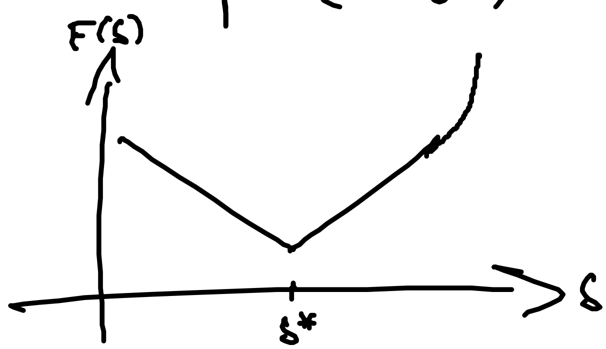


5.3 The set of all dual optimal solutions

F linear in the vicinity of δ => opt. dual solution p is gradient of F in point δ.

Def: Let $S \subseteq \mathbb{R}^n$ convex and $F: S \to \mathbb{R}$ convex and $\delta^* \in S$. Then p is a subgradient of F at δ^* if

$$F(\delta^*) + p^T \cdot (\delta - \delta^*) \leq F(\delta) \quad \forall \delta \in S$$



Remark: If F is linear (or at least differentiable) in δ^* then there is a unique subgradient, namely the gradient.
If δ^* is a breakpoint of F, then there are several subgradients.

Theorem: Suppose that the LP $\min c^T x$ s.t. $Ax = b^*$, $x \geq 0$ is feasible and bounded. Then p is an optimal solution to the dual LP if and only if p is a subgradient of F at b^*.

Proof: "$\Rightarrow$": Let p be an opt. dual solution

$\Rightarrow p^T \cdot b^* = F(b^*)$. For $b \in S$ and $x \in P(b)$

$$p^T \cdot b \leq c^T x \quad \text{(weak duality)}$$

$$\Rightarrow p^T \cdot b \leq F(b) = \min_{x \in P(b)} c^T x$$

$$\Rightarrow \quad F(b^*) - p^T b^* + p^T b \leq F(b)$$

"$\Leftarrow$": Assume that p subgradient of F at b^*

$$F(b^*) + p^T(b - b^*) \leq F(b) \quad \forall b \in S.$$

For some $x \geq 0$ let $b := A \cdot x \Rightarrow x \in P(b)$

$$\Rightarrow F(b) \leq c^T x$$

$$\Rightarrow p^T \cdot A \cdot x = p^T \cdot b \leq F(b) - F(b^*) + p^T \cdot b^*$$
$$\leq c^T x - F(b^*) + p^T \cdot b^*$$

$$\Rightarrow p^T A x \leq c^T x - F(b^*) + p^T b^* \quad \forall x \geq 0$$

$$\Rightarrow p^T A \leq c^T \quad \text{(i.e. } p \text{ is feasible dual sol.)}$$

For $x = 0$ we get $0 \leq -F(b^*) + p^T \cdot b^*$

$$\Rightarrow p^T \cdot b^* \geq F(b^*) \Rightarrow p \text{ optimal.} \square$$

5.4 Global dependence on the cost vector

Let $A \in \mathbb{R}^{m \times n}$, $b \in \mathbb{R}^m$ such that $\{x \mid Ax = b, x \geq 0\} \neq \phi$ and for $c \in \mathbb{R}^n$

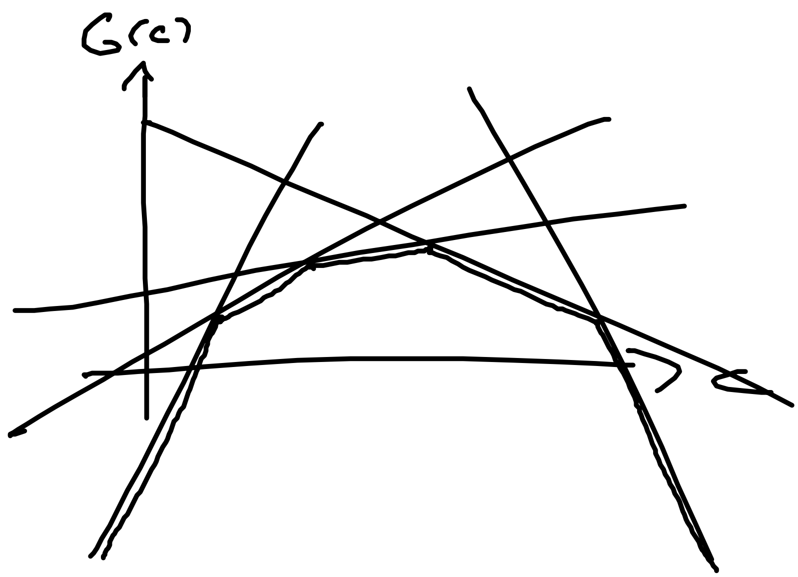
$$Q(c) := \{p \mid p^T A \leq c^T\}$$

$$T := \{c \mid Q(c) \neq \phi\} \quad \text{convex}$$

Note that $T := \{c \mid \underbrace{\min c^T x \text{ s.t. } Ax = b, x \geq 0}_{=: G(c)} \text{ is finite}\}$

$$G(c) = \min_{i=1,\dots,N} c^T \cdot x^i$$

where $x^1, \dots, x^N$ are the basic feasible solutions.



Theorem: For a feasible LP in standard form

(i) T is convex

(ii) $G(c)$ is a piecewise linear concave function of c.

(iii) If for some $c \in T$ the primal LP has a unique opt. sol. x^*, then G is linear in the vicinity of c and its gradient is equal to x^*.

5.5. Parametric programming.

Let $A \in \mathbb{R}^{m \times n}$, $b \in \mathbb{R}^m$ and $c, d \in \mathbb{R}^n$

Task: Solve

$$g(\Theta) := \min \ (c + \Theta \cdot d)^T \cdot x$$
$$\text{s.t.} \quad Ax = b$$
$$x \geq 0$$

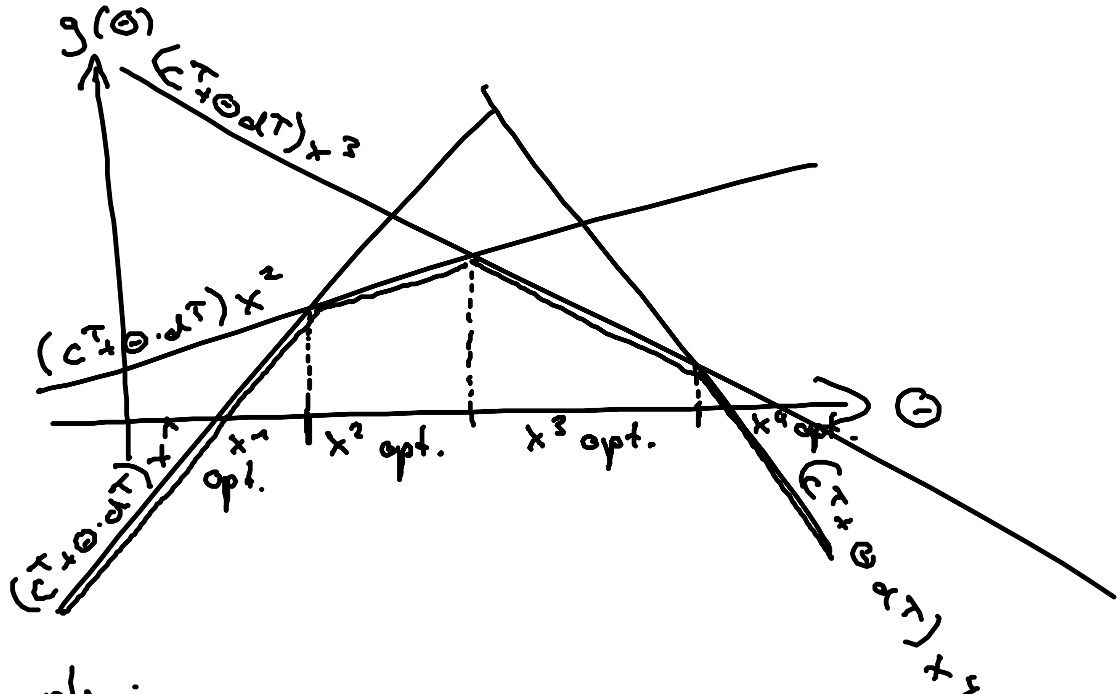
for all $\Theta \in \mathbb{R}$ with $g(\Theta) > -\infty$.

Assume that $\{x \mid Ax = b, x \geq 0\} \neq \emptyset$.

Notice that

$$g(\Theta) = \min_{i=1,\dots,N} (c^T + \Theta d^T) \cdot x^i$$

where $x^1, \dots, x^N$ are basic feas. sol.



Example:

$$\min \ (-3 + 2\Theta) x_1 + (3 - \Theta) \cdot x_2 + x_3$$
$$\text{s.t.} \quad x_1 + 2x_2 - 3x_3 \leq 5$$
$$2x_1 + x_2 - 4x_3 \leq 7$$
$$x_1, x_2, x_3 \geq 0$$

$\longrightarrow$ Tableau

	0	$-3+2\Theta$	$3-\Theta$	1	0	0
$x_4 =$	5	1	2	-3	1	0
$x_5 =$	7	2	1	-4	0	1

If $-3+2\Theta \geq 0$ and $3-\Theta \geq 0$, then this sol. is optimal

$$\Rightarrow g(\Theta) = 0 \quad \text{for } \Theta \in [\tfrac{3}{2}, 3]$$

Consider $\Theta > 3 \Rightarrow \bar{c}_2 < 0$ do pivoting step

$\longrightarrow$

	$-7.5+2.5\Theta$	$-4.5+2.5\Theta$	0	$5.5-1.5\Theta$	$-1.5+0.5\Theta$	0
$x_2 =$	2.5	0.5	1	-1.5	0.5	0
$x_5 =$	4.5	1.5	0	-2.5	-0.5	1

optimal for $3 \leq \Theta \leq \frac{5.5}{1.5} = \frac{11}{3}$

$$\Rightarrow g(\Theta) = 7.5 - 2.5 \cdot \Theta \quad \text{for } \Theta \in [3, \tfrac{11}{3}]$$

Consider $\Theta > \frac{11}{3} \Rightarrow \bar{c}_3 < 0$ and the LP is unbounded

$$g(\Theta) = -\infty \quad \text{for } \Theta \in (\tfrac{11}{3}, \infty)$$

Go back to initial tableau and consider $\Theta < \frac{3}{2} \Rightarrow \bar{c}_1 < 0$ do pivoting operation

$\longrightarrow$

$10.5 - 7 \cdot \Theta$	0	$4.5 - 2\Theta$	$-5 + 4\Theta$	0	$1.5 - \Theta$
$x_4 = 1.5$	0	1.5	-1	1	-0.5
$x_1 = 3.5$	1	0.5	-2	0	0.5

optimal for $\frac{5}{4} \leq \Theta \leq \frac{3}{2}$

$$g(\Theta) = -10.5 + 7 \cdot \Theta \quad \text{for } \Theta \in \left[\tfrac{5}{4}, \tfrac{3}{2}\right]$$

Consider $\Theta < \frac{5}{4} \Rightarrow \bar{c}_3 < 0$ and the LP is unbounded

$$g(\Theta) = -\infty \quad \text{for } \Theta \in \left(-\infty, \tfrac{5}{4}\right).$$

This approach works in general and terminates after a finite number of steps since no basis is visited more than once.
(avoid cycling by using, e.g., lexicographic rule.)