6.4 Dantzig-Wolfe decomposition

Consider an LP of the following form

$$\begin{aligned} \min\ & c_1^T x_1 + c_2^T x_2 \\ \text{s.t. } & D_1 x_1 + D_2 x_2 = b_0 \\ & F_1 \cdot x_1 = b_1 \\ & F_2 x_2 = b_2 \\ & x_1, x_2 \geq 0 \end{aligned}$$

with $c_1 \in \mathbb{R}^{n_1}$, $c_2 \in \mathbb{R}^{n_2}$, $b_0 \in \mathbb{R}^{m_0}$, $b_1 \in \mathbb{R}^{m_1}$, $b_2 \in \mathbb{R}^{m_2}$

Reformulation of the problem

For $i = 1,2$ let $P_i := \{x_i \geq 0 \mid F_i x_i = b_i\} \neq \emptyset$

$$\rightarrow \begin{aligned} \min\ & c_1^T x_1 + c_2^T x_2 \\ \text{s.t. } & D_1 x_1 + D_2 x_2 = b_0 \\ & x_1 \in P_1 \\ & x_2 \in P_2 \end{aligned}$$

Let x_i^j, $j \in J_i$, be the extreme points of P_i and w_i^k, $k \in K_i$, be a complete set of extreme rays of P_i, $i = 1,2$. Any $x_i \in P_i$ can be written as

$$x_i = \sum_{j \in J_i} \lambda_i^j \cdot x_i^j + \sum_{k \in K_i} \Theta_i^k \cdot w_i^k$$

with $\lambda_i^j, \Theta_i^k \geq 0$ and $\sum_{j \in J_i} \lambda_i^j = 1$, $i = 1,2$.

We define the <u>master problem</u> (reformulation of the given LP):

$$\min \sum_{j \in J_1} \lambda_1^j \cdot c_1^T \cdot x_1^j + \sum_{k \in K_1} \Theta_1^k \cdot c_1^T \cdot w_1^k$$

$$+ \sum_{j\in J_2} \lambda_2^j c_2^T x_2^j + \sum_{k\in K_2} \Theta_2^k \cdot c_2^T \cdot w_2^k$$

$$\text{s.t. } \sum_{j\in J_1} \lambda_1^j \cdot D_1 \cdot x_1^j + \sum_{k\in K_1} \Theta_1^k \cdot D_1 \cdot w_1^k$$

$$+ \sum_{j\in J_2} \lambda_2^j D_2 \cdot x_2^j + \sum_{k\in K_2} \Theta_2^k \cdot D_2 \cdot w_2^k = b_0$$

$$\sum_{j\in J_1} \lambda_1^j = 1$$

$$\sum_{j\in J_2} \lambda_2^j = 1$$

$$\lambda_i^j, \Theta_i^k \geq 0 \quad \forall i,j,k$$

Rewrite equality constraints in matrix form:

$$\sum_{j\in J_1} \lambda_1^j \cdot \begin{pmatrix} D_1 \cdot x_1^j \\ 1 \\ 0 \end{pmatrix} + \sum_{k\in K_1} \Theta_1^k \cdot \begin{pmatrix} D_1 \cdot w_1^k \\ 0 \\ 0 \end{pmatrix} + \sum_{j\in J_2} \lambda_2^j \cdot \begin{pmatrix} D_2 \cdot x_2^j \\ 0 \\ 1 \end{pmatrix} + \sum_{k\in K_2} \Theta_2^k \cdot \begin{pmatrix} D_2 w_2^k \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} b_0 \\ 1 \\ 1 \end{pmatrix}$$

The master problem has m_0+2 constraints (relatively few) but (in general) a huge number of variables/columns $\longrightarrow$ try to apply delayed column generation.

Let B be a feasible basis to the master problem and $p^T := c_B^T \cdot B^{-1}$ the associated dual solution.

$$p^T = (q^T, r_1, r_2) \text{ with } q \in \mathbb{R}^{m_0},\ r_1, r_2 \in \mathbb{R}$$

Compute reduced cost coefficient of a variable λ_1^j:

$$c_1^T \cdot x_1^j - (q^T, r_1, r_2) \cdot \begin{pmatrix} D_1 \cdot x_1^j \\ 1 \\ 0 \end{pmatrix} = (c_1^T - q^T \cdot D_1) x_1^j - r_1$$

Compute reduced cost coefficient of a variable Θ_1^k

$$c_1^T \cdot w_1^k - (q^T, r_1, r_2) \cdot \begin{pmatrix} D_1 w_1^k \\ 0 \\ 0 \end{pmatrix} = (c_1^T - q^T \cdot D_1) \cdot w_1^k$$

In order to solve the pricing problem (finding a variable λ_1^j or Θ_1^k with negative reduced cost) consider the following LP:

$$\min \ (c_1^T - q^T \cdot D_1) \cdot x_1 \quad \text{"the first subproblem"}$$
$$\text{s.t.} \qquad x_1 \in P_1$$

<u>Case 1:</u> First subproblem is unbounded

$\longrightarrow$ simplex algorithm yields an extreme ray w_1^k with $(c_1^T - q^T \cdot D_1) \cdot w_1^k < 0$

$\longrightarrow$ reduced cost of Θ_1^k is negative.

$\longrightarrow$ create column $\begin{pmatrix} D_1 \cdot w_1^k \\ 0 \\ 0 \end{pmatrix}$ and have it enter the basis in the master problem.

<u>Case 2:</u> First subproblem has an optimal solution of value $< r_1$

$\longrightarrow$ simplex method finds optimal extreme point x_1^j with $(c_1^T - p^T D_1) x_1^j < r_1$

$\longrightarrow$ reduced cost of variable λ_1^j is negative.

$\longrightarrow$ generate column $\begin{pmatrix} D_1 \cdot x_1^j \\ 1 \\ 0 \end{pmatrix}$ and have it enter the basis in the master problem.

Case 3: First subproblem has an optimal solution of value $\geq r_1$

$\longrightarrow (c_1^T - q^T D_1) x_1^j \geq r_1 \quad \forall j \in J_1$

$(c_1^T - q^T D_1) \cdot w_1^k \geq 0 \quad \forall k \in K_1$

$\longrightarrow$ Variables λ_1^j, Θ_1^k, $j \in J_1, k \in K_1$ have non-negative reduced cost coefficients.

Now we have solved the pricing problem for variables λ_1^j, Θ_1^k.

In order to solve the pricing problem for variables λ_2^j, Θ_2^k consider the second subproblem:

$$\min \ (c_2^T - q^T \cdot D_2) \cdot x_2$$
$$\text{s.t.} \quad x_2 \in P_2$$

and proceed as above.

Summary: Given problem is transformed to an equivalent problem (master problem) with few constraints but many variables. Pricing problem can be solved by solving smaller LPs.

Economic interpretation ... see book...

Generalization:
$$\min \sum_{i=1}^{t} c_i^T x_i$$
$$\text{s.t.} \quad \sum_{i=1}^{t} D_i x_i = b_0$$
$$F_i x_i = b_i \quad \forall i = 1, \dots, t$$

$x_i \geq 0$ "

Proceed as before $\rightarrow$ t subproblems for pricing.

Remark: Sometimes also useful for $t=1$.

Phase I of Dantzig-Wolfe:

Find feasible basis of master problem:

Determine extreme points of P_1 and P_2

$\rightarrow$ formulate an auxiliary master problem with slack variables and penalty costs in the objective function .... see book.