# Chapter 8: Complexity of linear progr. and the ellipsoid method

The Klee-Minty cubes show that the simplex has a quite bad worst-case behavior. In this chapter we discuss an efficient algorithm for linear prog.

## 8.1. Running time of algorithms

What is an "efficient" "algorithm"

running times

Turing machines or other formal model of computation

For us, an algorithm A consists of "elementary" steps, i.e., for example variable assignment and simple arithmetic operations which only take a constant amount of time.

bit model

arithmetic model

count bit operations (e.g. adding two bits etc.) adding two $n$-bit numbers takes $n+1$ steps; multiplying these numbers takes $O(n^2)$ steps.

simple arithmetic operations on arbitrary numbers can be performed in constant time.

The number of elementary steps of A depends on the input I, in particular on the size of I, denoted by size(I).

<u>bit model</u>

size(I) = # bits (in a predefined format)

<u>arithmetic model</u>

size(I) = # bits to encode structure of I + # numbers.

The running time of A is bounded by $f: \mathbb{N} \to \mathbb{N}$ if A performs at most $f(n)$ elementary steps on any input of size $\leq n$.

"worst case running time"

<u>Def</u> ("Efficient algorithm")

An algorithm runs in polynomial time, if its running time is bounded by a polynomial in the bit model. It even runs in strongly polynomial time, if its running time is bounded by a polynomial both in the bit model and the arithmetic model.

Remark: An algorithm that is polynomial in the arithmetic model and has the property that the size of any number produced in the course of this algorithm is bounded by a polynomial in the input size, runs in strongly polynomial time.

## 8.2 The key geometric result behind the ellipsoid method

Def: $D \in \mathbb{R}^{n \times n}$ symmetric is positive definite if $x^T \cdot D \cdot x > 0 \quad \forall x \in \mathbb{R}^n \setminus \{0\}$

Remark: The following are equivalent for a symmetric $D \in \mathbb{R}^{n \times n}$

(i) $D$ is positive definite

(ii) $D^{-1}$ is positive definite

(iii) $D$ has only real and positive eigenvalues.

(iv) $D = B^T \cdot B$ for a nonsingular matrix $B \in \mathbb{R}^{n \times n}$.

<u>Def:</u> A set $E \subseteq \mathbb{R}^n$ of the form

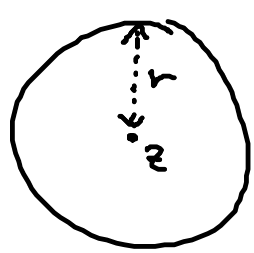
$$E = E(z, D) := \{x \in \mathbb{R}^n \mid (x-z)^T \cdot D^{-1} \cdot (x-z) \leq 1\}$$

where $D \in \mathbb{R}^{n \times n}$ is positive definite is called an <u>ellipsoid</u> with center $z$.

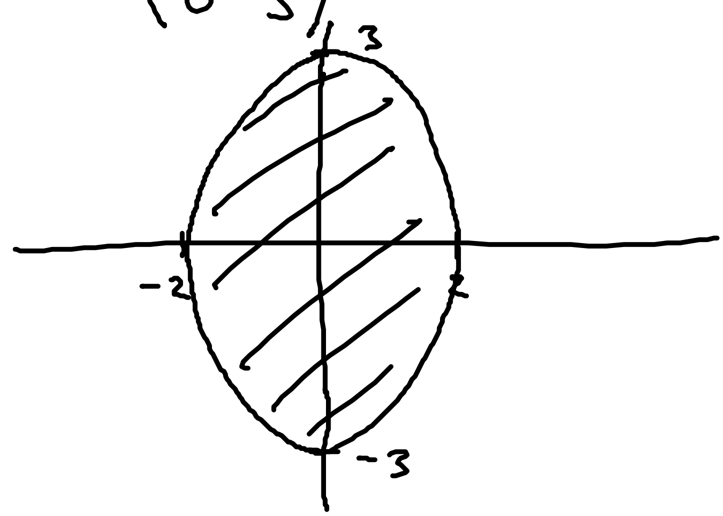
<u>Example:</u> (i) For $r > 0$, the ellipsoid

$$E(z, r^2 \cdot I) = \{x \in \mathbb{R}^n \mid (x-z)^T (x-z) \leq r^2\}$$
$$= \{x \in \mathbb{R}^n \mid \|x-z\|_2 \leq r\}$$

is the ball of radius $r$ centered at $z$.




(ii) $D = \begin{pmatrix} 4 & 0 \\ 0 & 9 \end{pmatrix}$ $z = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$



**Def:** If $D \in \mathbb{R}^{n \times n}$ is nonsingular and $\delta \in \mathbb{R}^n$ then the mapping $S: \mathbb{R}^n \to \mathbb{R}^n$ defined by $S(x) := Dx + \delta$ is an **affine transform**

For $L \subseteq \mathbb{R}^n$ let $S(L) := \{ y \in \mathbb{R}^n \mid y = Dx + \delta \text{ for some } x \in \mathbb{R}^n \}$

Define the volume of $L \subseteq \mathbb{R}^n$ as

$$\mathrm{Vol}(L) = \int_{x \in L} 1 \cdot dx$$

**Lemma:** If $S(x) = D \cdot x + \delta$ then

$$\mathrm{Vol}(S(L)) = |\det D| \cdot \mathrm{vol}(L).$$

**Proof:** $\mathrm{vol}(S(L)) = \int_{y \in S(L)} 1 \cdot dy$

$$= \int_{x\in L} |\det(J(x))|\,dx$$

where $J(x)$ is the Jacobian matrix associated with $S$, i.e.,

$$(J(x))_{ij} = \frac{\partial S_i(x)}{\partial x_j} = D_{ij}$$

$$= \int_{x\in L} |\det D|\,dx = |\det D| \operatorname{vol}(L).$$

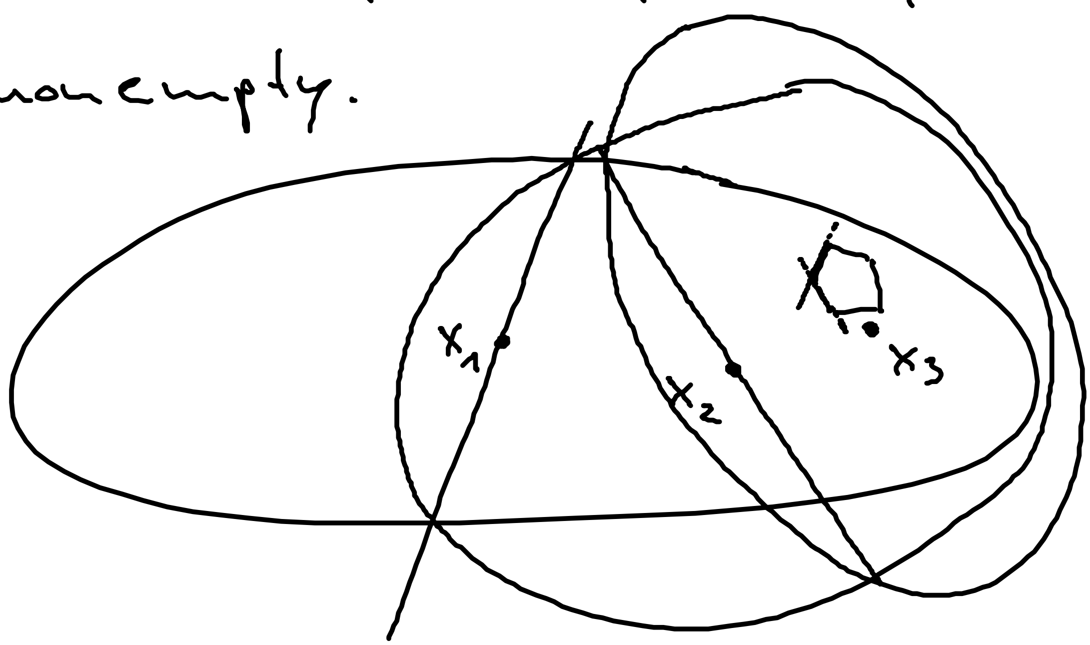
□

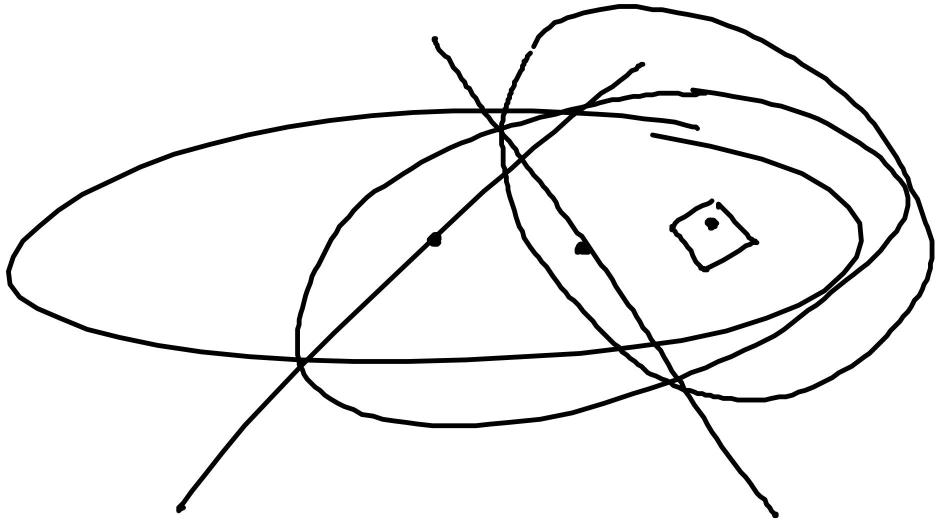
<u>Rough idea of the ellipsoid method:</u>

The ellipsoid method can be used to decide whether a given polyhedron

$$P = \{x \in \mathbb{R}^n \mid Ax \geq b\}$$

is nonempty.





The key insight for the ellipsoid method is that the volume of the ellipsoids strictly decreases in every iteration.