Chapter 7. Network Simplex

7.1. Basic definitions

Undirected Graphs. $G=(V,E)$ is an undirected graph if

$E \subseteq \binom{V}{2} = \{\{u,v\} \mid u,v \in V,\ u \neq v\}$. Nodes u,v are incident to the edge $\{u,v\}$. $\delta(v) = \{\{v,u\} \in E\}$, the degree of v is $\deg(v) = |\delta(v)|$.

A path is a sequence $v_1,\dots,v_k$ of nodes such that $\{v_i, v_{i+1}\} \in E$ $\forall i$ and $v_i \neq v_j$ for $i \neq j$.

A cycle is a sequence $v_1,\dots,v_k$ of nodes such that $v_1,\dots,v_{k-1}$ form a path and $v_k = v_1$ and $\{v_{k-1}, v_1\} \in E$ and $k \geq 4$.

An undirected graph is connected if there exits a path from every node to every other.

Directed Graphs. $G=(V,E)$ is a directed graph if $E \subseteq V \times V = \{(u,v) \mid u,v \in V,\ u \neq v\}$. For an edge (u,v) the node u is called the start node and v is the end node, and (u,v) is outgoing from u and incoming into v.

$\delta^+(v) = \{(v,u) \in E\}$, $\delta^-(v) = \{(u,v) \in E\}$.

For a directed graph construct the underlying undirected graph by ignoring the directions of the edges and delete possible „double edges".

A directed graph is connected if the underlying undirected graph is connected.

A $\left\{\begin{array}{c}\underline{\text{path}}\\ \underline{\text{cycle}}\end{array}\right\}$ in a directed graph is a sequence of nodes that forms a $\left\{\begin{array}{c}\text{path}\\ \text{cycle}\end{array}\right\}$ in the underlying undirected graph plus a sequence of connecting arcs between these nodes.

Exception: cycles of length 2 are allowed!

A path (or cycle) can have <u>forward</u> and <u>backward</u> edges.

A path with only forward arcs is called <u>directed</u>.

<u>Trees.</u> A <u>tree</u> is an undirected graph that is connected and has no cycles. A node of degree 1 in a tree is called a <u>leaf</u>.

<u>Theorem.</u>

(a) Every tree with more than one node has at least one leaf. (in fact: two leaves!)

(b) An undirected graph with n nodes is a tree if and only if it is connected and has $n-1$ edges.

(c) Given two nodes u, v in a tree, $u \neq v$, there exists a unique path from u to v.

(d) If an undirected graph G arises from a tree by adding one new edge, then G has exactly one cycle.

Let $G=(V,E)$ be a connected undirected graph. A tree $T=(V,E')$ with $E' \subseteq E$ is called a <u>spanning tree</u>.

<u>Theorem.</u> Let $G=(V,E)$ be a connected undirected graph and $E_0 \subseteq E$ a set of edges that forms no cycles. Then there exists a set $E' \subseteq E$, $E_0 \subseteq E'$ such that $T := (V, E')$ is a spanning tree of G.

7.2 The network flow problem

A *network* is a directed graph $G = (V, E)$ together with additional numerical information:

- $b_v \in \mathbb{R}$ $\quad \forall v \in V$ $\quad$ (in-flow at node v)
- $u_e \geq 0$ or $+\infty$ $\quad \forall e \in E$ $\quad$ (capacity of edge e)
- $c_e \in \mathbb{R}$ $\quad \forall e \in E$ $\quad$ (cost of edge e)

A node with $b_v > 0$ is called a *source* and a *sink* if $b_v < 0$.

In the network flow problem we are searching for a flow $(f_e)_{e \in E} \in \mathbb{R}^E$.

A vector $(f_e)_{e \in E}$ is a *feasible flow* if these constraints are satisfied:

$$\sum_{e \in \delta^+(v)} f_e - \sum_{e \in \delta^-(v)} f_e = b_v \qquad \forall v \in V \qquad \text{(flow conservation)}$$

$$f_e \leq u_e \qquad \forall e \in E$$

$$f_e \geq 0 \qquad \forall e \in E$$

Note:
$$\underbrace{\sum_{v \in V} \left(\sum_{e \in \delta^+(v)} f_e - \sum_{e \in \delta^-(v)} f_e \right)}_{\overset{\text{flow conservation}}{=} \ \sum_{v \in V} b_v} \overset{!}{=} \sum_{e \in E} f_e - f_e = 0$$

The *general minimal cost network flow problem*:

$$\min \sum_{e \in E} c_e f_e$$

$$\text{s.t.} \quad \sum_{e \in \delta^+(v)} f_e - \sum_{e \in \delta^-(v)} f_e = b_v \qquad \forall v \in V$$

$$\left.\begin{aligned} f_e &\le u_e \\ f_e &\ge 0 \end{aligned}\right\} \forall e \in E$$

A linear program in standard form if $u_e = \infty \quad \forall e \in E$.

(<u>uncapacitated network</u>)

Special cases:

- shortest path
- max. flow
- transportation problem (equivalent!)
- assignment problem

Variants (equivalent)

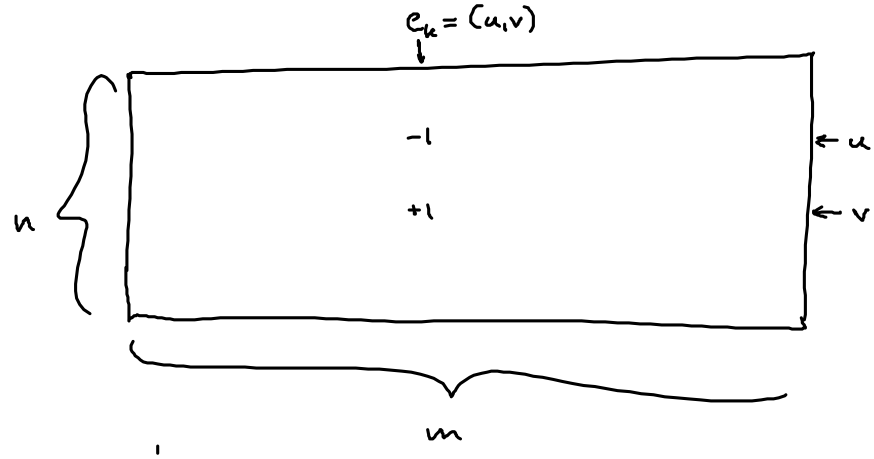
- transportation problem
- problem with only one source and one sink
- ——— " ——— no ——— " ——— no — " — (circulation problem)
- network with node capacities
- network with lower bounds on the edge flows: $f_e \ge d_e$

<u>Matrix formulation.</u>

W.l.o.g. assume $V = \{1, \dots, n\}$ and $|E| = m \ge n-1$, edges somehow ordered: $E = \{e_1, \dots, e_m\}$

Incidence matrix: $A = (a_{i,k})_{\substack{1 \le i \le n \\ 1 \le k \le m}}$ with

$$a_{i,k} := \begin{cases} +1 & \text{if } e_k = (i, v) \quad \text{for some } v \in V \\ -1 & \text{if } e_k = (v, i) \quad \text{——— " ———} \\ 0 & \text{otherwise} \end{cases}$$



If $f = (f_{e_1}, \ldots, f_{e_m})^T$ is a flow vector then flow conservation states

$$A \cdot f = b$$

The rows of A add up to the zero vector $\Rightarrow$ rank $A \le n-1$

A flow vector f with $A \cdot f = 0$ is called a circulation.