Chapter 7. Network Simplex

7.1. Basic definitions

Undirected Graphs. $G=(V,E)$ is an undirected graph if $E \subseteq \binom{V}{2} = \{\{u,v\} \mid u,v \in V,\ u \neq v\}$. Nodes u,v are incident to the edge $\{u,v\}$. $\delta(v) = \{\{v,u\} \in E\}$, the degree of v is $\deg(v) = |\delta(v)|$.

A path is a sequence $v_1, \dots, v_k$ of nodes such that $\{v_i, v_{i+1}\} \in E$ $\forall i$ and $v_i \neq v_j$ for $i \neq j$.

A cycle is a sequence $v_1, \dots, v_k$ of nodes such that $v_1, \dots, v_{k-1}$ form a path and $v_k = v_1$ and $\{v_{k-1}, v_1\} \in E$ and $k \geq 4$.

An undirected graph is connected if there exits a path from every node to every other.

Directed Graphs. $G=(V,E)$ is a directed graph if $E \subseteq V \times V = \{(u,v) \mid u,v \in V,\ u \neq v\}$. For an edge (u,v) the node u is called the start node and v is the end node, and (u,v) is outgoing from u and incoming into v.

$\delta^+(v) = \{(v,u) \in E\}$, $\delta^-(v) = \{(u,v) \in E\}$.

For a directed graph construct the underlying undirected graph by ignoring the directions of the edges and delete possible „double edges".

A directed graph is connected if the underlying undirected graph is connected.

A $\begin{Bmatrix} \text{path} \\ \text{cycle} \end{Bmatrix}$ in a directed graph is a sequence of nodes that forms

a $\begin{Bmatrix} \text{path} \\ \text{cycle} \end{Bmatrix}$ in the underlying undirected graph plus a sequence of connecting arcs between these nodes.

Exception: cycles of length 2 are allowed!

A path (or cycle) can have forward and backward edges.

A path with only forward arcs is called directed.

Trees. A tree is an undirected graph that is connected and has no cycles. A node of degree 1 in a tree is called a leaf.

Theorem.

(a) Every tree with more than one node has at least one leaf. (in fact: two leaves!)

(b) An undirected graph with n nodes is a tree if and only if it is connected and has $n-1$ edges.

(c) Given two nodes u, v in a tree, $u \neq v$, there exists a unique path from u to v.

(d) If an undirected graph G arises from a tree by adding one new edge, then G has exactly one cycle.

Let $G = (V, E)$ be a connected undirected graph. A tree $T = (V, E')$ with $E' \subseteq E$ is called a spanning tree.

Theorem. Let $G = (V, E)$ be a connected undirected graph and $E_0 \subseteq E$ a set of edges that forms no cycles. Then there exists a set $E' \subseteq E$, $E_0 \subseteq E'$ such that $T := (V, E')$ is a spanning tree of G.

7.2 The network flow problem

A <u>network</u> is a directed graph $G=(V,E)$ together with additional numerical information:

- $b_v \in \mathbb{R}$ $\quad \forall v \in V$ $\quad$ (inflow at node v)
- $u_e \geq 0$ or $+\infty$ $\quad \forall e \in E$ $\quad$ (capacity of edge e)
- $c_e \in \mathbb{R}$ $\quad \forall e \in E$ $\quad$ (cost of edge e)

A node with $b_v > 0$ is called a <u>source</u> and a <u>sink</u> if $b_v < 0$.

In the network flow problem we are searching for a flow $(f_e)_{e\in E} \in \mathbb{R}^E$.

A vector $(f_e)_{e\in E}$ is a <u>feasible flow</u> if these constraints are satisfied:

$$\sum_{e\in\delta^+(v)} f_e - \sum_{e\in\delta^-(v)} f_e = b_v \qquad \forall v \in V \qquad \text{(flow conservation)}$$

$$f_e \leq u_e \qquad \forall e \in E$$

$$f_e \geq 0 \qquad \forall e \in E$$

Note: $$\sum_{v\in V}\Big(\sum_{e\in\delta^+(v)} f_e - \sum_{e\in\delta^-(v)} f_e\Big) \overset{!}{=} \sum_{e\in E} f_e - f_e = 0$$

$$\Vert \leftarrow \text{flow conservation}$$

$$\sum_{v\in V} b_v$$

The <u>general minimal cost network flow problem</u>:

$$\min \sum_{e\in E} c_e f_e$$

$$\text{s.t.} \quad \sum_{e\in\delta^+(v)} f_e - \sum_{e\in\delta^-(v)} f_e = b_v \qquad \forall v \in V$$

$$\left.\begin{aligned} f_e &\le u_e \\ f_e &\ge 0 \end{aligned}\right\} \forall e \in E$$

A linear program in standard form if $u_e = \infty \;\; \forall e \in E$.
(uncapacitated network)

Special cases:

- shortest path
- max. flow
- transportation problem (equivalent!)
- assignment problem

Variants (equivalent)

- transportation problem
- problem with only one source and one sink
- ———— " ——— no ——— " ——— no — " — (circulation problem)
- network with node capacities
- network with lower bounds on the edge flows: $f_e \ge d_e$

Matrix formulation.

W.l.o.g. assume $V = \{1, \ldots, n\}$ and $|E| = m \ge n-1$, edges somehow ordered: $E = \{e_1, \ldots, e_m\}$

Incidence matrix: $A = (a_{i,k})_{\substack{1 \le i \le n \\ 1 \le k \le m}}$ with

$$a_{i,k} := \begin{cases} +1 & \text{if } e_k = (i, v) \text{ for some } v \in V \\ -1 & \text{if } e_k = (v, i) \quad \text{———— " ————} \\ 0 & \text{otherwise} \end{cases}$$

$$
n\left\{\vphantom{\begin{array}{c}1\\1\\1\\1\\1\end{array}}\right.
\underbrace{\begin{array}{c}
\qquad\qquad e_k=(u,v)\qquad\qquad\\
\left[\begin{array}{ccc}
\quad & & \quad \\
 & -1 & \\
 & +1 & \\
 & &
\end{array}\right]
\end{array}}_{m}
\begin{array}{l}
\\
\\
\leftarrow u\\
\leftarrow v\\
\\
\end{array}
$$

If $f = (f_{e_1}, \dots, f_{e_m})^T$ is a flow vector then flow conservation states

$$A \cdot f = b$$

The rows of A add up to the zero vector $\Rightarrow$ rank $A \leq n-1$

A flow vector f with $A \cdot f = 0$ is called a circulation.