Cycles & Circulations

A flow vector f with $A \cdot f = 0$ is called a circulation.

Let C be a cycle and F the set of forward arcs and B the set of backward arcs of C. Then

$$h_e^C := \begin{cases} +1 & \forall e \in F \\ -1 & \forall e \in B \\ 0 & \text{otherwise} \end{cases}$$

is the simple circulation associated with C.

The cost of the cycle C is

$$c^T \cdot h^C = \sum_{e \in F} c_e - \sum_{e \in B} c_e$$

For $\Theta \in \mathbb{R}$ we push a flow of Θ around C if we replace a flow f by the flow $f + \Theta \cdot h^C$. Then the cost changes to

$$c^T \cdot (f + \Theta h^C) - c^T f = \Theta \cdot c^T h^C.$$

7.3 Network simplex algorithm

Consider the uncapacitated network flow problem

$$\begin{aligned} \min \quad & c^T f \\ \text{s.t.} \quad & A \cdot f = b \\ & f \geq 0 \end{aligned}$$

$$|V| = n, \quad |E| = m \Rightarrow A \in \{-1, 0, 1\}^{n \times m}$$

Assume:

(a) $\sum b_v = 0$

$v \in V$

(b) G is connected

The truncated constraints matrix $\tilde{A}$ arises from A by deleting the last row; accordingly let $\tilde{b} := (b_1, \dots, b_{n-1})^T$.

Basic solutions.

Definition. (& Theorem) Let $T \subseteq E$ be a set of $n-1$ edges whose underlying undirected $n-1$ edges form a tree.

Let $f_e = 0$ for all $e \notin T$. Then the values of f_e for $e \in T$ can be uniquely determined by the linear equations $\tilde{A} \cdot f = \tilde{b}$. Such a flow vector is called a tree solution.
If additionally $f \geq 0$ it is a feasible tree solution.

Procedure to obtain a tree solution (efficienty):

- Take node n as the root node of the tree
- Start with the leaves
- Proceed the tree upwards

Let B be the submatrix of $\tilde{A}$ consisting of all columns corresponding to tree edges. $\Rightarrow B = \{-1, 0, +1\}^{(n-1)\times(n-1)}$

Theorem. $\det B \in \{-1, +1\}$.

Proof: Reorder nodes $\leftrightarrow$ reordering of the rows of B
Reorder edges $\leftrightarrow$ —— " —— columns —"—

Then the reordered matrix is a lower triagular matrix with ± 1 on the diagonal. $\square$.

<u>Corollary.</u> Tree solutions are uniquely determined and $\tilde{A}$ has rank $n-1$.

<u>Theorem.</u> f is a tree solution $\Leftrightarrow$ f is a basic solution.

<u>Corollary.</u>

(a) For every basis matrix B, the inverse B^{-1} has only integer entries.

(b) If b has only integer entries, then every basis solution has only integer entries and there exists an integer optimal sol.

(c) If c has only integer entries, then every dual basic solution has only integer entries and there exists an integer optimal dual solution.

<u>Basis change.</u>

Choose an edge $(u,v) \in E \setminus T$

$\Rightarrow$ (u,v) together with some edges in T define a unique cycle C

Choose an orientation of C such that (u,v) is a forward edge

Let F and B be the set of forward and backward edges.

Push a flow of $\theta \geq 0$ through C:

$$f_e \rightsquigarrow \hat{f}_e = \begin{cases} f_e + \Theta & \text{if } e \in F \\ f_e - \Theta & \text{if } e \in B \\ f_e & \text{otherwise} \end{cases}$$

(maintain flow conservation!)

to maintain nonnegativity:

$$\Theta \le \Theta^* := \min_{e \in B} f_e \quad \text{if } B \neq \emptyset$$

If $B = \emptyset$ (i.e. the cycle is directed) we can choose $\Theta = \infty$.

An edge e that attains the above minimum gets $\hat{f}_e = 0$ and exits the tree.

<u>NOTE:</u> degeneracy – if $\Theta^* = 0$ the basis changes, but not the flow.

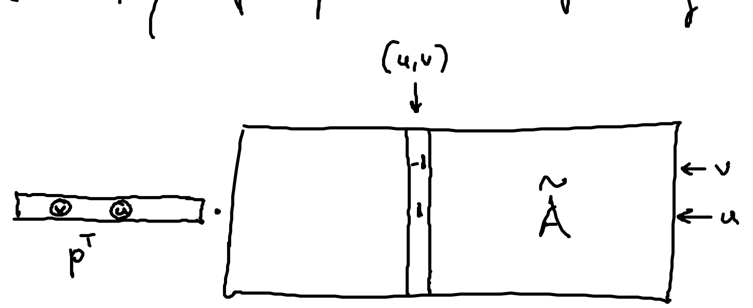
cost change: $\Theta^* \cdot \underbrace{\left(\sum_{e \in F} c_e - \sum_{e \in B} c_e \right)}_{\bar{c}_{(u,v)} \, !!}$

$$\bar{c}^T = c^T - p^T \cdot \tilde{A} \quad \text{with} \quad p^T = c_B^T \cdot B^{-1}$$

$\longrightarrow p \in \mathbb{R}^{n-1}$, $p_v \leftrightarrow$ node v

For an edge (u,v), we need the entry of $p^T \cdot \tilde{A}$ corresponding to (u,v)

$$\bar{c}_{(u,v)} = c_{(u,v)} - \begin{cases} p_u - p_v & \text{if } u,v \neq n \\ p_u & \text{if } v = n \\ -p_v & \text{if } u = n \end{cases}$$



with $p_n = 0$: $\quad \overline{c}_{(u,v)} = c_{(u,v)} - (p_u - p_v) \quad \forall (u,v) \in E$

For an edge in the basis, reduced cost must be 0

$$\Rightarrow \quad p_u - p_v = c_{(u,v)} \quad \forall (u,v) \in T$$
$$p_n = 0$$

Procedure:

- start at the root node, where $p_n = 0$
- proceed T downwards: $p_u = c_{(u,v)} + p_v$ if u is beneath v
  $p_v = -c_{(u,v)} + p_u$ if v is beneath u

<u>Iteration of the network simplex algorithm:</u>

Given a basic feasible (tree) solution f with a tree T

(1) Compute the dual vector p as above

(2) Compute the reduced costs $\overline{c}_{(u,v)} = c_{(u,v)} - (p_u - p_v)$ for all edges $(u,v) \notin T$.

If $\overline{c}_{(u,v)} \geq 0 \quad \forall (u,v) \notin T$ then STOP (f is optimal).

Else choose some edge (u,v) with $\overline{c}_{(u,v)} < 0$.

(3) (u,v) forms a unique cycle C with the edges in T.

Orient C such that (u,v) is a forward arc and let B be the set of backward arcs.

If $B \neq \emptyset$ then STOP (optimal cost is $-\infty$).

(4) Let $\Theta^* = \min_{e \in B} f_e$ and push Θ^* units of flow around C, updating f accordingly.

Add (u,v) to the basis and remove one of the edges for which the minimum in (*) is attained.