

## Cycles & Circulations

A flow vector  $f$  with  $A \cdot f = 0$  is called a circulation.

Let  $C$  be a cycle and  $F$  the set of forward arcs and  $B$  the set of backward arcs of  $C$ . Then

$$h_e^C := \begin{cases} +1 & \forall e \in F \\ -1 & \forall e \in B \\ 0 & \text{otherwise} \end{cases} \quad \text{is the simple circulation associated with } C.$$

The cost of the cycle  $C$  is

$$c^T \cdot h^C = \sum_{e \in F} c_e - \sum_{e \in B} c_e$$

For  $\theta \in \mathbb{R}$  we push a flow of  $\theta$  around  $C$  if we replace a flow  $f$  by the flow  $f + \theta \cdot h^C$ . Then the cost changes to

$$c^T \cdot (f + \theta h^C) - c^T \cdot f = \theta \cdot c^T \cdot h^C.$$

### 7.3 Network simplex algorithm

Consider the uncapacitated network flow problem

$$\begin{array}{ll} \min & c^T f \\ \text{s.t.} & A \cdot f = b \\ & f \geq 0 \end{array}$$

$$|V| = n, |E| = m \Rightarrow A \in \{-1, 0, 1\}^{n \times m}$$

Assume:

$$(a) \sum b_v = 0$$

(b)  $G$  is connected

The truncated constraints matrix  $\tilde{A}$  arises from  $A$  by deleting the last row; accordingly let  $\tilde{b} = (b_1, \dots, b_{m-1})^T$ .

Basic solutions.

Definition. Let  $T \subseteq E$  be a set of  $n-1$  edges whose underlying undirected  <sup>$n-1$</sup>  edges form a tree.  
(& Theorem)

Let  $f_e = 0$  for all  $e \notin T$ . Then the values of  $f_e$  for  $e \in T$  can be uniquely determined by the linear equations

$\tilde{A} \cdot f = \tilde{b}$ . Such a flow vector is called a tree solution.  
If additionally  $f \geq 0$  it is a feasible tree solution.

Procedure to obtain a tree solution (efficiently):

- Take node  $n$  as the root node of the tree
- Start with the leaves
- Proceed the tree upwards

Let  $B$  be the submatrix of  $\tilde{A}$  consisting of all columns corresponding to tree edges.  $\Rightarrow B = \{-1, 0, +1\}^{(n-1) \times (n-1)}$

Theorem.  $\det B \in \{-1, +1\}$ .

Proof: Reorder nodes  $\iff$  reordering of the rows of  $B$   
 Reorder edges  $\iff$   $\text{---} n \text{---}$  columns  $\text{---}$

Then the reordered matrix is a lower triangular matrix with  $\pm 1$  on the diagonal.  $\square$

Corollary. Tree solutions are uniquely determined and  $\tilde{A}$  has rank  $n-1$ .

Theorem.  $f$  is a tree solution  $\Leftrightarrow f$  is a basic solution.

Corollary.

(a) For every basis matrix  $B$ , the inverse  $B^{-1}$  has only integer entries.

(b) If  $b$  has only integer entries, then every basic solution has only integer entries and there exists an integer optimal sol.

(c) If  $c$  has only integer entries, then every dual basic solution has only integer entries and there exists an integer optimal dual solution.

Basis change.

Choose an edge  $(u,v) \in E \setminus T$

$\Rightarrow (u,v)$  together with some edges in  $T$  define a unique cycle  $C$

Choose an orientation of  $C$  such that  $(u,v)$  is a forward edge

Let  $F$  and  $B$  be the set of forward and backward edges.

Push a flow of  $\theta \geq 0$  through  $C$ :

$$f_e \rightsquigarrow \hat{f}_e = \begin{cases} f_e + \theta & \text{if } e \in F \\ f_e - \theta & \text{if } e \in B \\ f_e & \text{otherwise} \end{cases} \quad (\text{maintain flow conservation!})$$

to maintain nonnegativity:

$$\theta \leq \theta^* := \min_{e \in B} f_e \quad \text{if } B \neq \emptyset$$

If  $B = \emptyset$  (i.e. the cycle is directed) we can choose  $\theta = \infty$ .

An edge  $e$  that attains the above minimum gets  $\hat{f}_e = 0$  and exits the tree.

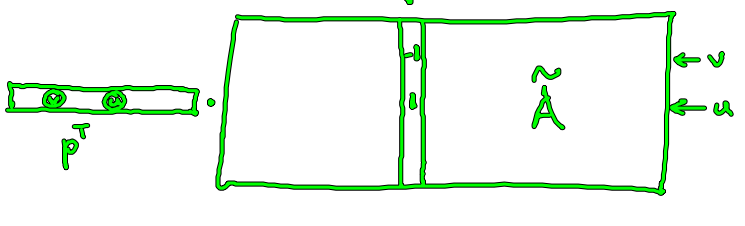
NOTE: degeneracy — if  $\theta^* = 0$  the basis changes, but not the flow.

$$\text{cost change: } \theta^* \cdot \underbrace{\left( \sum_{e \in F} c_e - \sum_{e \in B} c_e \right)}_{\bar{c}_{(u,v)}!!}$$

$$\bar{c}^T = c^T - p^T \cdot \tilde{A} \quad \text{with } p^T = c_B^T \cdot B^{-1}$$

$$\rightarrow p \in \mathbb{R}^{n-1}, \quad p_v \leftrightarrow \text{node } v$$

For an edge  $(u,v)$ , we need the entry of  $p^T \cdot \tilde{A}$  corresponding to  $(u,v)$

$$\bar{c}_{(u,v)} = c_{(u,v)} - \begin{cases} p_u - p_v & \text{if } u,v \neq n \\ p_u & \text{if } v = n \\ -p_v & \text{if } u = n \end{cases}$$


$$\text{with } p_u = 0 : \quad \bar{c}_{(u,v)} = c_{(u,v)} - (p_u - p_v) \quad \forall (u,v) \in E$$

For an edge in the basis, reduced cost must be 0

$$\begin{aligned} \Rightarrow \quad p_u - p_v &= c_{(u,v)} & \forall (u,v) \in T \\ p_u &= 0 \end{aligned}$$

Procedure:

- start at the root node, where  $p_u = 0$
- proceed  $T$  downwards:  $p_u = c_{(u,v)} + p_v$  if  $u$  is beneath  $v$   
 $p_v = -c_{(u,v)} + p_u$  if  $v$  is beneath  $u$

Iteration of the network simplex algorithm:

Given a basic feasible (tree) solution  $f$  with a tree  $T$

- ① Compute the dual vector  $p$  as above
- ② Compute the reduced costs  $\bar{c}_{(u,v)} = c_{(u,v)} - (p_u - p_v)$  for all edges  $(u,v) \notin T$ .  
If  $\bar{c}_{(u,v)} \geq 0 \quad \forall (u,v) \notin T$  then STOP ( $f$  is optimal).  
Else choose some edge  $(u,v)$  with  $\bar{c}_{(u,v)} < 0$ .
- ③  $(u,v)$  forms a unique cycle  $C$  with the edges in  $T$ .  
Orient  $C$  such that  $(u,v)$  is a forward arc and let  $B$  be the set of backward arcs.  
If  $B \neq \emptyset$  then STOP (optimal cost is  $-\infty$ ).

④ Let  $\theta^* = \min_{e \in B} f_e$  and push  $\theta^*$  units of flow around  $C$ , updating  $f$  accordingly.

Add  $(u,v)$  to the basis and remove one of the edges for which the minimum in ④ is attained.