Chapter 3: Interior point methods

3.1. The affine scaling algorithm

Consider a pair of LPs : $A \in \mathbb{R}^{m \times n}$, $c \in \mathbb{R}^n$, $b \in \mathbb{R}^m$

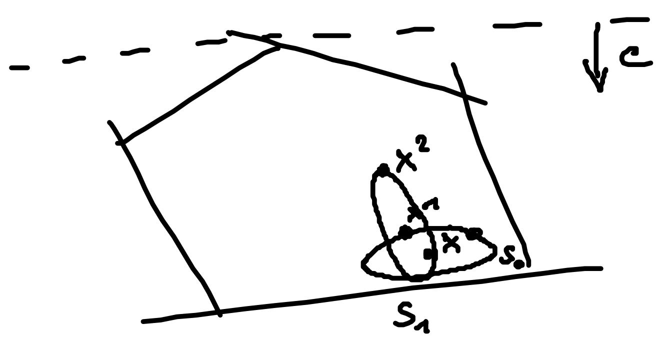
$$\min \ c^T x \qquad\qquad \max \ p^T b$$
$$\text{s.t.} \ \boxed{\begin{aligned} Ax &= b \\ x &\geq 0 \end{aligned}} \qquad\qquad \text{s.t.} \ p^T A \leq c^T$$

Let $P = \{x \in \mathbb{R}^n \mid Ax = b, x \geq 0\}$ and

$\{x \in P \mid x > 0\}$ the <u>interior of P</u>. A point $x \in P$ with $x > 0$ is called <u>interior point of P</u>.

<u>Main idea:</u>

- start with interior point x^0
- Form ellipsoid S_0 centered at x^0 and contained in the interior of P.
- Optimize $c^T x$ over all $x \in S_0$ and find opt. sol. x^1
- Form ellipsoid S_1 centered at x^1 ....



<u>Lemma:</u> Let $\beta \in (0,1)$, $y \in \mathbb{R}^n$, $y > 0$ and

$$S := \{x \in \mathbb{R}^n \mid \sum_{i=1}^n \frac{(x_i - y_i)^2}{y_i^2} \leq \beta^2\}$$

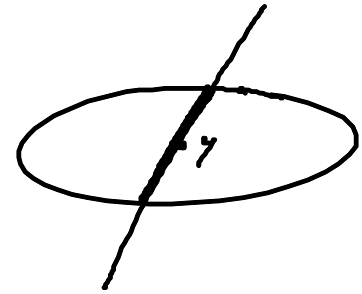
Then $x > 0$ for all $x \in S$.

<u>Proof:</u> $x \in S$, $i \in \{1, \dots, n\}$:

$$(x_i - y_i)^2 \leq \beta^2 \cdot y_i^2 \implies |x_i - y_i| < y_i$$
$$\implies -x_i + y_i < y_i \implies x_i > 0. \quad \square$$

Fix some $y \in \mathbb{R}^n$ with $y > 0$ and $Ay = b$.

$$Y := \operatorname{diag}(y_1, \dots, y_n)$$



Then:

$$x \in S \iff \|Y^{-1} \cdot (x-y)\| \le \beta$$

↑ Euclidean norm.

$$\iff (x-y)^T \cdot \left(Y^{-1} \cdot Y^{-1}\right)(x-y) \le \beta^2$$

In particular, S is an ellipsoid centered at y.

Let $S_0 := \{x \in S \mid Ax = b\}$ "section" of ellipsoid S.

S_0 is itself an ellipsoid contained in the interior of P.

Optimize $c^T x$ over all $x \in S_0$

$$\begin{aligned} \min\ & c^T x \\ \text{s.t.}\ & Ax = b \\ & \|Y^{-1} \cdot (x-y)\| \le \beta \end{aligned}$$

Reformulate this by setting $d := x - y$:

$$\begin{aligned} \min\ & c^T \cdot d \\ \text{s.t.}\ & Ad = 0 \\ & \|Y^{-1} \cdot d\| \le \beta \end{aligned}$$

<u>Lemma:</u> Assume that rows of A are linearly independent and c is not a linear combination of rows of A. An optimal solution d^* is given by

$$d^* := -\beta \cdot \frac{Y^2 \cdot (c - A^T \cdot p)}{\|Y \cdot (c - A^T \cdot p)\|}$$

where $p := (A \cdot Y^2 \cdot A^T)^{-1} \cdot A \cdot Y^2 \cdot c$.

Moreover, $x := y + d^* \in P$ and

$$c^T x = c^T y - \beta \cdot \|Y \cdot (c - A^T p)\| < c^T y.$$

Proof: see book. $\square$

Remarks:

(i) If $d^* \geq 0$, the LP is unbounded since $A \cdot d^* = 0$ and $y + \alpha \cdot d^* > 0$ for $\alpha > 0$ and $c^T \cdot d^* < 0$.

(ii) Assume that y is a nondegenerate basic feasible sol. (contradiction to $y > 0$). Let B be the corresponding basic matrix and $A = (B, N)$ (w.l.o.g.)

Let $Y = \mathrm{diag}(y_1, \dots, y_m, 0, \dots, 0)$ and

$Y_0 = \mathrm{diag}(y_1, \dots, y_m)$

Then $A \cdot Y = (B, N) \cdot \left(\begin{array}{c|c} Y_0 & 0 \\ \hline 0 & 0 \end{array}\right) = (B \cdot Y_0, 0)$ and

$$p = (A Y^2 \cdot A^T)^{-1} \cdot A \cdot Y^2 \cdot c$$
$$= (B^T)^{-1} \cdot Y_0^{-2} \cdot B^{-1} \cdot B \cdot Y_0^2 \cdot c_B = (B^T)^{-1} \cdot c_B$$

is the corresponding dual basic solution.

The vector p corresponding to an arbitrary primal solution y is called <u>dual estimate</u>, even if y is not basic.

The vector $r := c - A^T p$ becomes

$$r = c - A^T \cdot (B^T)^{-1} \cdot c_B$$

the reduced cost vector.

If y is degenerate, the matrix $A \cdot Y^2 \cdot A^T$ is singular and this interpretation breaks down.

(iii) If $r = c - A^T \cdot p \geq 0$, then p is a dual feasible solution and $r^T \cdot y = (c - A^T \cdot p)^T \cdot y = c^T y - p^T \cdot A \cdot y = c^T y - p^T \cdot b$ is the difference between the primal and dual objective function value (called the <u>duality gap</u>).

If $r^T y = 0$ then complementary slackness cond. hold and y and p are both optimal.

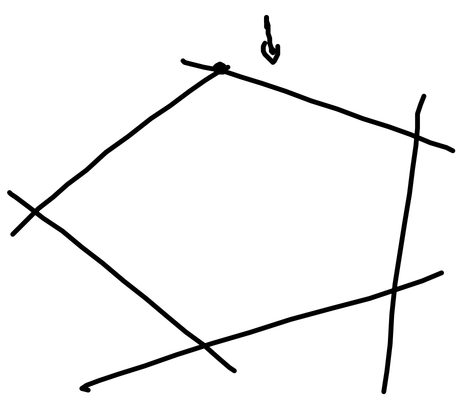
The affine scaling algorithm

1) Start with a feasible $x^0 > 0$; set $k := 0$
2) Let $X_k = \mathrm{diag}(x_1^k, \dots, x_n^k)$

$$p^k = (A \cdot X_k^2 \cdot A^T)^{-1} A \cdot X_k^2 \cdot c$$

$$r^k = c - A^T \cdot p$$

3) If $r^k \geq 0$ and $(r^k)^T \cdot x^k < \varepsilon$, then stop; x^k is primal ε-opt. and p^k is dual ε-opt.
4) If $-X_k^2 \cdot r^k \geq 0$, then stop; the optimal cost is $-\infty$.
5) Let $x^{k+1} = x^k - \beta \cdot \dfrac{X_k^2 \cdot r^k}{\| X_k \cdot r^k \|}$; $k := k+1$; goto 2)



Convergence

Assumptions:

(i) rows of A are linearly indep.
(ii) c is not a linear combin. of rows of A.
(iii) There exists an opt. sol.
(iv) There exists a positive feasible sol.
(v) every basic feasible solution to the primal LP is non-deg.
(vi) At every basic feasible sol. to the primal LP, the reduced costs of non-basic variables are non-zero.

Theorem: Under these assumptions, for $\varepsilon = 0$, the algorithm converges to a pair of primal and dual opt. solutions.

Initialization: Consider auxiliary problem:

$$\begin{aligned} \min\ & c^T x + M \cdot x_{n+1} \\ \text{s.t.}\ & A \cdot x + (b - A \cdot \mathbb{1}) \cdot x_{n+1} = b \\ & (x, x_{n+1}) \geq 0 \end{aligned}$$

and notice that $(\mathbb{1}, 1)$ is a positive feasible solution.

Computational performance:

The running time of one iteration of the algorithm is dominated by the computation of p. This takes $O(m^2 \cdot n)$ steps.