

Chapter 3: Interior point methods

3.1. The affine scaling algorithm

Consider a pair of LPs : $A \in \mathbb{R}^{m \times n}$, $c \in \mathbb{R}^n$, $b \in \mathbb{R}^m$

$$\min c^T x$$

$$\text{s.t. } \boxed{\begin{array}{l} Ax = b \\ x \geq 0 \end{array}}$$

$$\max p^T b$$

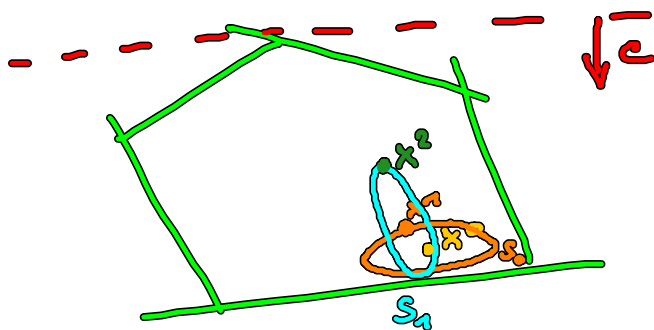
$$\text{s.t. } p^T A \leq c^T$$

Let $P = \{x \in \mathbb{R}^n \mid Ax = b, x \geq 0\}$ and

$\{x \in P \mid x > 0\}$ the interior of P. A point $x \in P$ with $x > 0$ is called interior point of P.

Main idea:

- start with interior point x^0
- Form ellipsoid S_0 centered at x^0 and contained in the interior of P .
- Optimize $c^T x$ over all $x \in S_0$ and find opt. sol. x^1
- Form ellipsoid S_1 centered at x^1



Lemma: Let $\beta \in (0, 1)$, $\gamma \in \mathbb{R}^n$, $\gamma > 0$ and

$$S := \left\{ x \in \mathbb{R}^n \mid \sum_{i=1}^n \frac{(x_i - \gamma_i)^2}{\gamma_i^2} \leq \beta^2 \right\}$$

Then $x > 0$ for all $x \in S$.

Proof: $x \in S$, $i \in \{1, \dots, n\}$:

$$(x_i - \gamma_i)^2 \leq \beta^2 \cdot \gamma_i^2 \Rightarrow |x_i - \gamma_i| < \gamma_i$$

$$\Rightarrow -x_i + \gamma_i < \gamma_i \Rightarrow x_i > 0. \quad \square$$

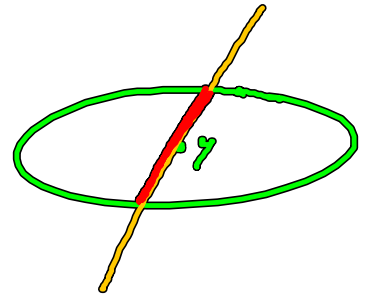
Fix some $y \in \mathbb{R}^n$ with $y \succ 0$ and $Ay = b$.

$$Y := \text{diag}(y_1, \dots, y_n)$$

Then:

$$x \in S \iff \|Y^{-1} \cdot (x - y)\| \leq \beta$$

\uparrow
Euclidean norm.



$$\iff (x - y)^T \cdot (Y^{-1} \cdot Y^{-1}) \cdot (x - y) \leq \beta^2$$

In particular, S is an ellipsoid centered at y.

Let $S_0 := \{x \in S \mid Ax = b\}$ "section" of ellipsoid S.

S_0 is itself an ellipsoid contained in the interior of P.

Optimize $c^T x$ over all $x \in S_0$.

$$\begin{aligned} \min c^T x \\ \text{s.t. } Ax = b \end{aligned}$$

$$\|Y^{-1} \cdot (x - y)\| \leq \beta$$

Reformulate this by setting $d := x - y$:

$$\begin{aligned} \min c^T d \\ \text{s.t. } Ad = 0 \end{aligned}$$

$$\|Y^{-1} \cdot d\| \leq \beta$$

Lemma: Assume that rows of A are linearly independent and c is not a linear combination of rows of A. An optimal solution d^* is given by

$$d^* := -\beta \cdot \frac{Y^2 \cdot (c - A^T \cdot p)}{\|Y \cdot (c - A^T \cdot p)\|}$$

$$\text{where } p := (A \cdot Y^2 \cdot A^T)^{-1} \cdot A \cdot Y^2 \cdot c.$$

Moreover, $x := y + d^* \in P$ and

$$c^T x = c^T y - \beta \cdot \|Y \cdot (c - A^T p)\| < c^T y.$$

Proof: see book. \square

Remarks:

(i) If $d^* \geq 0$, then LP is unbounded since $A \cdot d^* = 0$ and $\gamma + \alpha \cdot d^* > 0$ for $\alpha > 0$ and $c^T \cdot d^* < 0$.

(ii) Assume that γ is a nondegenerate basic feasible sol. (contradiction to $\gamma > 0$). Let B be the corresponding basic matrix and $A = (B, N)$ (w.l.o.g.)

Let $Y = \text{diag}(\gamma_1, \dots, \gamma_m, 0, \dots, 0)$ and

$$Y_0 = \text{diag}(\gamma_1, \dots, \gamma_m)$$

Then $A \cdot Y = (B, N) \cdot \begin{pmatrix} Y_0 & 0 \\ 0 & 0 \end{pmatrix} = (B \cdot Y_0, 0)$ and

$$p = (A Y^2 \cdot A^T)^{-1} \cdot A \cdot Y^2 \cdot c$$

$$= (B^T)^{-1} \cdot Y_0^{-2} \cdot B^{-1} \cdot B \cdot Y_0^2 \cdot c_B = (B^T)^{-1} \cdot c_B$$

is the corresponding dual basic solution.

The vector p corresponding to an arbitrary primal solution γ is called dual estimate, even if γ is not basic.

The vector $r := c - A^T p$ becomes

$$r = c - A^T \cdot (B^T)^{-1} \cdot c_B$$

the reduced cost vector.

If γ is degenerate, the matrix $A \cdot Y^2 \cdot A^T$ is singular and this interpretation breaks down.

(iii) If $r = c - A^T \cdot p \geq 0$, then p is a dual feasible solution and $r^T \cdot \gamma = (c - A^T \cdot p)^T \cdot \gamma = c^T \gamma - p^T \cdot A \cdot \gamma = c^T \gamma - p^T \cdot b$ is the difference between the primal and dual objective function value (called the duality gap).

If $r^k = 0$ the complementary slackness cond. hold and y and p are both optimal.

The affine scaling algorithm

1) Start with a feasible $x^0 > 0$; set $k := 0$

2) Let $X_k = \text{diag}(x_1^k, \dots, x_n^k)$

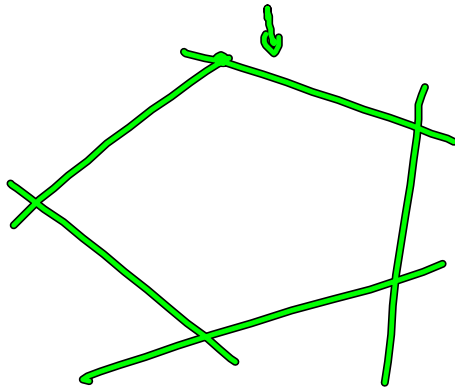
$$p^k = (A \cdot X_k^2 \cdot A^T)^{-1} A \cdot X_k^2 \cdot c$$

$$r^k = c - A^T \cdot p$$

3) If $r^k \geq 0$ and $(r^k)^T \cdot x^k \leq \epsilon$, then stop; x^k is primal ϵ -opt. and p^k is dual ϵ -opt.

4) If $-X_k^2 \cdot r^k \geq 0$, then stop; the optimal cost is $-\infty$.

5) Let $x^{k+1} = x^k - \rho \cdot \frac{X_k^2 \cdot r^k}{\|X_k \cdot r^k\|}$; $k := k+1$; goto 2)



Convergence

Assumptions:

(i) rows of A are linearly indep.

(ii) c is not a linear combin. of rows of A .

(iii) There exists an opt. sol.

(iv) There exists a positive feasible sol.

(v) every basic feasible solution to the primal LP is non-deg.

(vi) At every basic feasible sol. to the primal LP, the reduced costs of non-basic variables are non-zero.

Theorem: Under these assumptions, for $\epsilon = 0$, the algorithm converges to a pair of primal and dual opt. solutions.

Initialization: Consider auxiliary problem:

$$\min c^T x + M \cdot x_{\text{aux}}$$

$$\text{s.t. } A \cdot x + (b - A \cdot \mathbf{1}) \cdot x_{\text{aux}} = b$$

$$(x, x_{\text{aux}}) \geq 0$$

and notice that $(\mathbf{1}, 1)$ is a positive feasible solution.

Computational performance:

The running time of one iteration of the algorithm is dominated by the computation of p . This takes $O(n^2 \cdot n)$ steps.