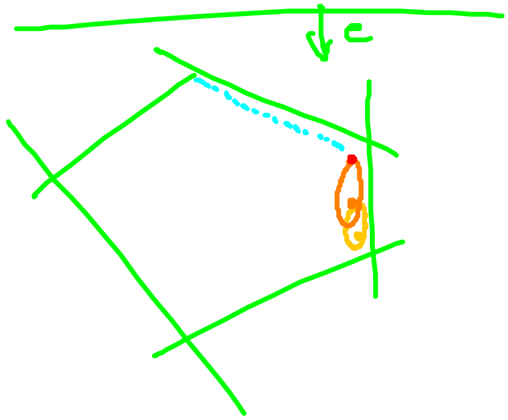


9.3 The potential reduction algorithm

$$\begin{array}{llll} \min & c^T x & \max & p^T b \\ \text{s.t.} & Ax = b & \text{s.t.} & p^T A + s^T = c^T \\ & x \geq 0 & & s \geq 0 \end{array}$$

Assume that rows of A are linearly independent and there exist $x^0 > 0$ and (p^0, s^0) with $s^0 > 0$ which are feasible for primal and dual LP, respectively.

Main new idea: Stay away from the boundary.

Potential function:

$$G(x,s) = q \cdot \log s^T x - \sum_{j=1}^{n} \log x_j - \sum_{j=1}^{n} \log s_j$$

where $q > n$ is a constant.

If x and (p,s) are feasible, then

$$c^T x - b^T \cdot p = (s^T + p^T A) \cdot x - x^T \cdot A^T \cdot p = s^T x \quad \text{"duality gap"}$$

Theorem: An algorithm that maintains primal and dual feasibility and reduces $G(x,s)$ by at least $\delta > 0$ at each iteration, finds solutions with duality gap $(s^K)^T \cdot x^K \le \varepsilon$ after

$$K = \left\lceil \frac{G(x^0, s^0) + (q-n) \cdot \log 1/\varepsilon - n \cdot \log n}{\delta} \right\rceil$$

iterations.

Proof: $G(x,s) = q \cdot \log s^T x - \sum_{j=1}^{n} \log x_j - \sum_{j=1}^{n} \log s_j$

$$= \underbrace{n \cdot \log s^T x - \sum_{j=1}^{n} \log(x_j \cdot s_j)}_{\text{minimum attained for}} + (q-n) \cdot \log s^T x$$

minimum attained for

$x_j \cdot s_j = \frac{s^T \cdot x}{n} \quad \forall j = 1, \dots, n$

$\ge n \cdot \log n + (q-n) \cdot \log s^T x$

$\Rightarrow n \cdot \log s^T x - \sum_{j=1}^{n} \log x_j - \sum_{j=1}^{n} \log s_j \ge n \cdot \log n \quad (*)$

Suppose that $G(x^{k+1}, s^{k+1}) - G(x^k, s^k) \le -\delta \; \forall k$

$\Rightarrow G(x^K, s^K) - G(x^0, s^0) \le -K \cdot \delta$

Our choice of K yields:

$G(x^K, s^K) \le -(q-n) \cdot \log 1/\varepsilon + n \log n$

Using def. of $G(x,s)$ and $(*)$ yields:

$G(x^K, s^K) \ge n \cdot \log n + (q-n) \cdot \log (s^K)^T \cdot x^K$

$(s^K)^T \cdot x^K \le \varepsilon$. $\square$

$\longrightarrow$ "potential reduction" algorithm.

<u>Idea:</u> Starting with feasible $x > 0$ and (p, s) with $s > 0$, try to find direction d such that

$$G(x+d, s) < G(x, s)$$

such that $A \cdot d = 0$, $\| X^{-1} \cdot d \| \leq \beta < 1$.

Minimizing $G(x+d, s)$ s.t. $Ad = 0$, $\| X^{-1} \cdot d \| \leq \beta$ is difficult due to objective function.

Idea: Linearize the objective function by taking the first order Taylor series expansion in d.

$$\min \left(\nabla_x G(x, s) \right)^T \cdot d$$
$$\text{s.t. } A \cdot d = 0$$
$$\| X^{-1} \cdot d \| \leq \beta$$

(Same result as in 9.1. except for different objective function $\hat{c}$)

$$\hat{c}_i = \frac{\partial G(x, s)}{\partial x_i} = \frac{q \cdot s_i}{s^T x} - \frac{1}{x_i}$$

Applying lemma from Sect. 9.1, we obtain opt. sol.

$$d^* = -\beta \cdot X \cdot \frac{u}{\| u \|}$$

where

$$u = X \cdot \left(\hat{c} - A^T \cdot (A X^2 A^T)^{-1} A X^2 \hat{c} \right)$$

Since $X\cdot\hat{e} = \frac{q}{s^T x} X\cdot s - \mathbb{1}$, we get

$$u = \left(I - XA^T(AX^2A^T)^{-1}AX\right)\left(\frac{q}{s^T x}Xs - \mathbb{1}\right)$$

Moreover, $G(x,s)$ decreases by $\underbrace{\beta\cdot\|u\|}_{\text{Lemma in 9.1.}} + \underbrace{O(\beta^2)}_{\text{higher order terms.}}$

$\Rightarrow$ If $\|u\|$ is large enough, then $G(x,s)$ decreases by at least $\delta > 0$. Otherwise update dual variables (omit details).

The potential reduction algorithm

1) Start with feasible $x^0 > 0$, (p^0, s^0) with $s^0 > 0$; set $k := 0$

2) If $(s^k)^T\cdot x^k < \varepsilon$ stop.

3) Let $X_k = \operatorname{diag}(x_1^k, \dots, x_n^k)$

$$\bar{A}_k = (A\cdot X_k)^T (A X_k^2 A^T)^{-1}\cdot A\cdot X_k$$

$$u^k = (I - \bar{A}_k)\cdot\left(\frac{q}{(s^k)^T x^k} X_k s - \mathbb{1}\right)$$

$$d^k = -\beta\cdot X_k\cdot u^k / \|u^k\|.$$

4) (Primal step) If $\|u^k\| \geq \gamma$, then

$$x^{k+1} := x^k + d^k, \quad s^{k+1} = s^k, \quad p^{k+1} = p^k$$

5) (Dual step) If $\|u^k\| < \gamma$, then

$$x^{k+1} = x^k, \quad s^{k+1} = \frac{(s^k)^T \cdot x^k}{q} (X_k)^{-1} \cdot (u^k + \mathbb{1})$$

$$p^{k+1} = p^k + (A X_k^2 A^T)^{-1} A X_k \left(X_k s - \frac{(s^k)^T x^k}{q} \cdot \mathbb{1} \right)$$

6) $k := k+1$, goto 2.

<u>Theorem:</u> Let $\beta < 1$, $\gamma < 1$

(i) If $\|u^k\| \geq \gamma$ then

$$G(x^{k+1}, s^{k+1}) - G(x^k, s^k) \leq -\beta \cdot \gamma + \frac{\beta^2}{2(1-\beta)}$$

(ii) If $\|u^k\| < \gamma$ then

$$G(x^{k+1}, s^{k+1}) - G(x^k, s^k) \leq -(q-n) + n \cdot \log \frac{q}{n} + \frac{\gamma^2}{2(1-\gamma)}$$

(iii) If $q = n + \sqrt{n}$, $\beta \approx 0.285$, $\gamma \approx 0.479$, then $G(x, s)$ decreases by at least $\delta = 0.079$ at each iteration.

<u>Proof:</u> see book. □

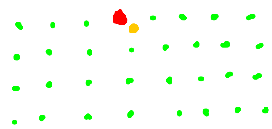
<u>Initialization:</u> see book (auxiliary problem ....)

<u>Complexity of the algorithm:</u>

The potential reduction algorithm finds ε-opt. solutions in time

$$O(n^{3.5} \cdot \log 1/\varepsilon + n^5 \cdot \log(n \cdot U))$$

where $U = \max\{|a_{ij}|, |b_i|, |c_j|\}$ (all integer)



If ε is taken sufficiently small, an optimal solution can be found by rounding.