Primal path following algorithm

$$\begin{array}{ll} \min & c^T x \\ \text{s.t.} & Ax = b \\ & x \geq 0 \end{array} \qquad \begin{array}{ll} \max & p^T b \\ \text{s.t.} & p^T A + s^T = c^T \\ & s \geq 0 \end{array}$$

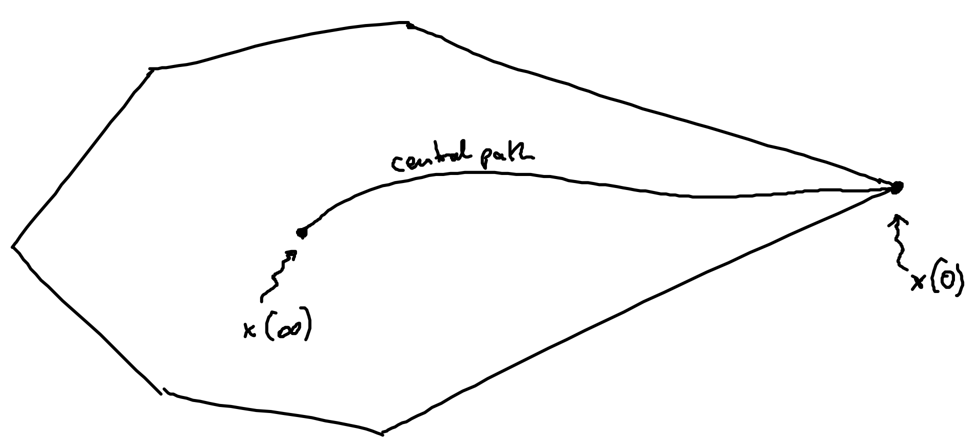
- Step 1: Ignore the inequality constraints and modify the objective function $\longrightarrow$ barrier function : $\mu > 0$

$$B_\mu(x) := \begin{cases} c^T x - \mu \sum_{j=1}^{n} \log x_j & \text{for } x > 0 \\ \infty & \text{if } x_j \leq 0 \text{ for some } j \end{cases}$$

$\longrightarrow$ barrier problem : $\quad \begin{array}{ll} \min & B_\mu(x) \\ \text{s.t.} & Ax = b \end{array}$

Assume all barrier problems for $\mu > 0$ have an optimal solution $x(\mu)$.

The <u>central path</u> is the set $\{x(\mu) \mid \mu > 0\}$



- $\lim_{\mu \to 0} x(\mu)$ is the optimal solution to the original LP
- $\lim_{\mu \to \infty} x(\mu)$ is called the analytic center

Ex.: ① $\min \; x$

s.t. $x \geq 0$

$\longrightarrow$ barrier function: $B_\mu(x) = x - \mu \log x \qquad (x > 0)$

$\longrightarrow$ minimize: $\frac{d}{dx} B_\mu(x) = 1 - \frac{\mu}{x} = 0 \iff x = \mu$

$\Rightarrow x(\mu) = \mu$

$\longrightarrow \lim_{\mu \to 0} x(\mu) = 0$

② $\min \; x_2$

s.t. $x_1 + x_2 + x_3 = 1$

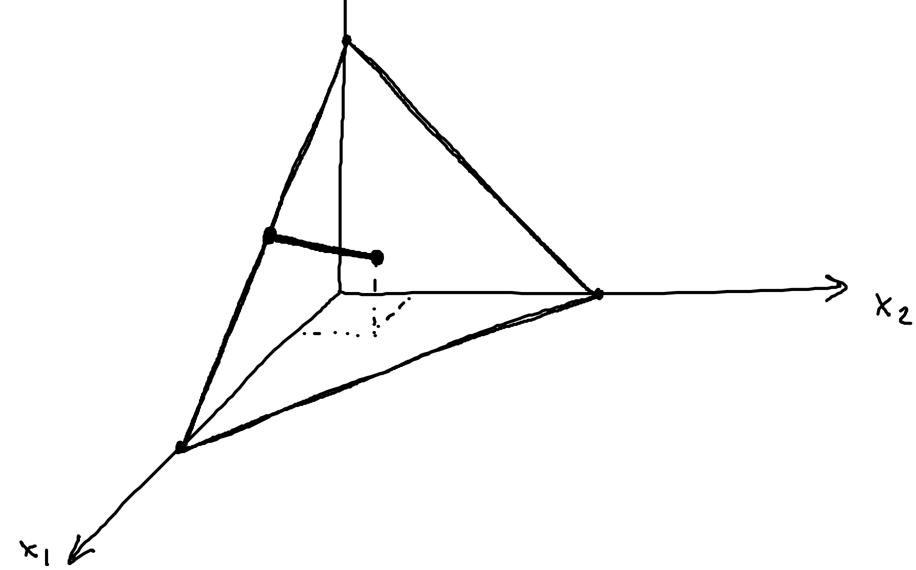
$x_i \geq 0$

$\longrightarrow$ barrier problem: $\min \; x_2 - \mu \log x_1 - \mu \log x_2 - \mu \log x_3$

s.t. $x_1 + x_2 + x_3 = 1$

⋮

$\longrightarrow$ central path: $x(\mu) = \frac{1}{2} \begin{pmatrix} \sqrt{9\mu^2 + 2\mu + 1} - 3\mu \\ 2 + 6\mu - 2\sqrt{9\mu^2 + 2\mu + 1} \\ \sqrt{9\mu^2 + 2\mu + 1} - 3\mu \end{pmatrix}$



$$\lim_{\mu \to 0} x(\mu) = \begin{pmatrix} 1/2 \\ 0 \\ 1/2 \end{pmatrix} \qquad \lim_{\mu \to \infty} x(\mu) = \begin{pmatrix} 1/3 \\ 1/3 \\ 1/3 \end{pmatrix}$$

Dual barier problem:
$$\max \; p^T b + \mu \sum_{j=1}^{n} \log s_j$$
$$\text{s.t. } p^T A + s^T = c^T$$

<u>Lemma.</u> (Karush-Kuhn-Tucker conditions)

$x(\mu)$, $p(\mu)$, $s(\mu)$ are optimal solutions to the primal and dual barier problems w.r.t. μ if and only if the following conditions hold:

$$A x(\mu) = b \qquad \text{with } X(\mu) = \begin{pmatrix} x_1(\mu) & & \\ & \ddots & \\ & & x_n(\mu) \end{pmatrix}$$
$$x(\mu) \geq 0$$
$$A^T p(\mu) + s(\mu) = c \qquad S(\mu) = \begin{pmatrix} s_1(\mu) & & \\ & \ddots & \\ & & s_n(\mu) \end{pmatrix}$$
$$s(\mu) \geq 0$$
$$X(\mu) S(\mu) e = \mu e \qquad e = (1, \dots, 1)$$

$$x_j(\mu) \cdot s_j(\mu) = \mu \quad \forall j$$

- Step 2: Approximate the barrier function by its Taylor series:

$$B_\mu(x+d) \approx B_\mu(x) + \sum_{i=1}^{n} \underbrace{\frac{\partial}{\partial x_i} B_\mu(x)}_{c_i - \frac{\mu}{x_i}} \cdot d_i$$

$$+ \frac{1}{2} \sum_{i,j=1}^{n} \underbrace{\frac{\partial^2}{\partial x_i \partial x_j} B_\mu(x)}_{\substack{0 \text{ if } i \neq j \\ \frac{\mu}{x_i^2} \text{ if } i=j}} \cdot d_i \cdot d_j$$

$$= B_\mu(x) + (c^T - \mu e X^{-1}) \cdot d + \frac{1}{2} \mu d^T \cdot (X^2)^{-1} \cdot d$$

$$\longrightarrow \min \quad (c^T - \mu e X^{-1}) \cdot d + \frac{1}{2} \mu d^T \cdot (X^2)^{-1} \cdot d$$

$$\text{s.t.} \quad A \cdot d = 0$$

- Step 3: Solve the approximation with Lagrangean multipliers:

Lagrangean function:

$$L(d,p) = (c^T - \mu e X^{-1}) d + \frac{1}{2} \mu d^T (X^2)^{-1} d - p^T \cdot A d$$

with

$$\frac{\partial}{\partial d_j} L(d,p) = 0 \quad \forall j$$

$$\frac{\partial}{\partial p_i} L(d,p) = 0 \quad \forall i$$

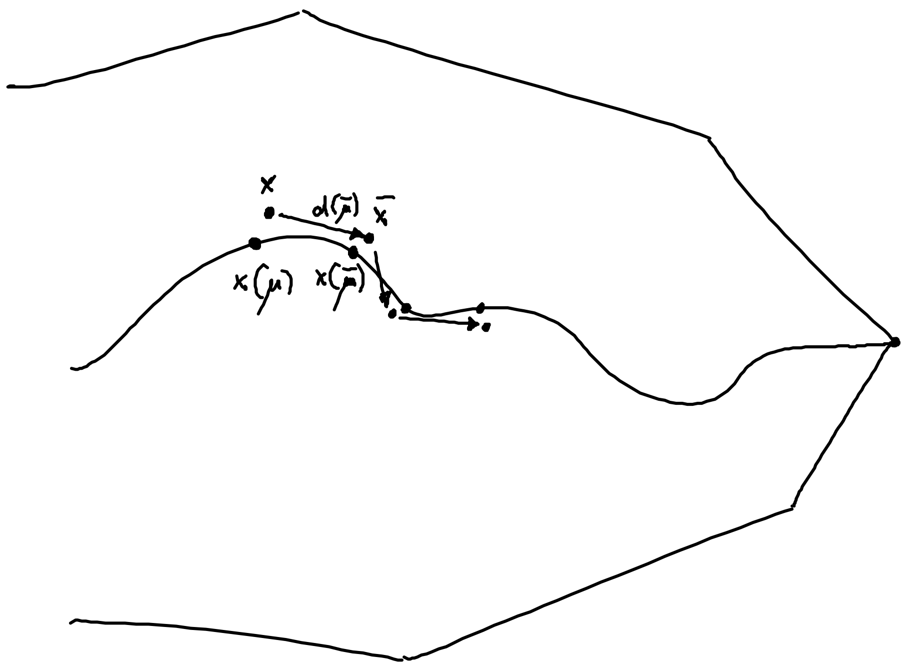
$\longrightarrow$ leads to a linear system:

$$\begin{pmatrix} \mu (X^2)^{-1} & -A^T \\ A & 0 \end{pmatrix} \cdot \begin{pmatrix} d \\ p \end{pmatrix} = \begin{pmatrix} \mu X^{-1} e - c \\ 0 \end{pmatrix}$$

solving this equation for d and p leads to

$d(\mu) \leftarrow$ Newton direction
$p(\mu)$

Newton step: given x, p, s, μ
new primal solution: $\bar{x} = x + d(\mu)$
new dual solution: $p(\mu)$, $c - A^T p(\mu)$
decrease μ to $\bar{\mu} : \alpha\mu$ with $0 < \alpha < 1$ (fixed)



- Algorithm:
Input: A, b, c, $\varepsilon > 0$ (optimality tolerance), $0 < \alpha < 1$, μ^0
initial primal / dual feasible solutions $x^0 > 0$, $s^0 > 0$, p^0

1. $k := 0$
2. If $(s^k)^T x^k < \varepsilon$ then STOP: current solution is ε-optimal
3. $$X_k := \begin{pmatrix} x_1^k & & \\ & \ddots & \\ & & x_n^k \end{pmatrix}, \quad \mu^{k+1} := \alpha \mu^k$$
4. Solve the linear system

$$\begin{pmatrix} \mu^{k+1}(X_k^2)^{-1} & -A^T \\ A & 0 \end{pmatrix} \begin{pmatrix} d \\ p \end{pmatrix} = \begin{pmatrix} \mu^{k+1} X_k^{-1} e - c \\ 0 \end{pmatrix}$$

for d and p

5. $x^{k+1} := x^k + d$, $p^{k+1} := p$, $s^{k+1} := c - A^T p$
6. $k := k+1$, go to 2.