

Theorem. Let $0 < \beta < 1$, $\mu^0 > 0$, $\epsilon > 0$.

If the primal path following method is used with parameter $\alpha = 1 - \frac{\sqrt{\beta} - \beta}{\sqrt{\beta} + \beta}$ and initial primal/dual feasible solution $x^0 > 0$, $s^0 > 0$, p^0 that satisfy

$$\left\| \frac{1}{\mu^0} X_0 S_0 e - e \right\| \leq \beta, \text{ then a primal/dual}$$

feasible solution (x^k, p^k, s^k) with duality gap

$(s^k)^T x^k \leq \epsilon$ is found after

$$K := \left\lceil \frac{\sqrt{\beta} + \beta}{\sqrt{\beta} - \beta} \log \left(\frac{(s^0)^T x^0}{\epsilon} \cdot \frac{1 + \beta}{1 - \beta} \right) \right\rceil = \left\lceil \frac{1}{1 - \alpha} \log \left(\frac{\epsilon_0}{\epsilon} \cdot \frac{1 + \beta}{1 - \beta} \right) \right\rceil$$

iterations.

→ solve an auxiliary LP with μ^0 , x^0 , s^0 such that

$$\left\| \frac{1}{\mu^0} X_0 S_0 e - e \right\| = \frac{1}{4}$$

→ # of iterations to reduce the duality gap from

$\epsilon_0 = (s^0)^T x^0$ to ϵ is

$$\left\lceil (2 + \frac{1}{4}\sqrt{n}) \log \left(\frac{\epsilon_0}{\epsilon} \cdot \frac{5}{3} \right) \right\rceil = O \left(\sqrt{n} \log \frac{\epsilon_0}{\epsilon} \right)$$

→ polynomial in n

with $O((n+n)^3)$ operations in one iteration

$$\parallel \\ O(n^3)$$

Primal-dual path following algorithm

Idea: Directly try to solve the KKT conditions:

$$Ax(\mu) = b$$

$$x(\mu) \geq 0$$

$$A^T p(\mu) + s(\mu) = c$$

$$s(\mu) \geq 0$$

$$X(\mu)S(\mu)e = e\mu$$

► Newton's method:

Given $F: \mathbb{R}^r \rightarrow \mathbb{R}^r$, problem: find a z^* such that

$$F(z^*) = 0$$

Assume we have an approximation z for z^*

→ find a direction d such that $z+d$ is a better approx.

Taylor series: $F(z+d) \approx F(z) + J_F(z) \cdot d$

with the Jacobian $J_F(z) = \left(\frac{\partial}{\partial z_i} F_i(z) \right)_{1 \leq i, j \leq n}$

$$F(z+d) = 0 \Leftrightarrow F(z) + J_F(z) \cdot d = 0$$

► Here:

$$F(x, p, s) = \begin{pmatrix} Ax - b \\ A^T p + s - c \\ XSe - \mu e \end{pmatrix}$$

→ solve the linear system

$$\begin{pmatrix} A & 0 & 0 \\ 0 & A^T & I \\ S & 0 & X \end{pmatrix} \cdot \begin{pmatrix} d_x \\ d_p \\ d_s \end{pmatrix} = -F(x, p, s)$$

$$(*) = \begin{pmatrix} 0 \\ 0 \\ \mu e - XSe \end{pmatrix}$$

► Updating the solution and the barrier parameter:

new solution: $x + \beta_{\text{ind}} d_x$, $p + \beta_{\text{ind}} d_p$, $s + \beta_{\text{ind}} d_s$

$$\text{with } \beta_{\text{primal}} = \min \left\{ 1, \alpha \cdot \min_{\substack{i \\ \text{such that} \\ (d_x)_i < 0}} \left(-\frac{x_i}{(d_x)_i} \right) \right\} \quad (**)$$

$$\beta_{\text{dual}} = \min \left\{ 1, \alpha \cdot \min_{\substack{i \\ \text{such that} \\ (d_s)_i < 0}} \left(-\frac{s_i}{(d_s)_i} \right) \right\}$$

where $0 < \alpha < 1$.

$$\bar{\mu} := \rho \cdot \frac{s^T x}{n} \quad \text{with } 0 < \rho \leq 1 \quad (***)$$

► Algorithm:

Input: $A, b, c, \varepsilon > 0, 0 < \alpha < 1, \mu^0$
 initial primal & dual feasible solutions $x^0 > 0, s^0 > 0, p^0$

1. $k := 0$

2. If $(s^k)^T x^k < \varepsilon$ then STOP: solution is ε -optimal.

3. Let $\rho^k \in (0, 1]$, update μ^k according to (***)

Solve the linear system (*) for d_x^k, d_p^k, d_s^k

4. Find the step lengths β_{primal} and β_{dual} according to (**)

$$5. x^{k+1} := x^k + \beta_{\text{primal}} \cdot d_x^k,$$

$$p^{k+1} := p^k + \beta_{\text{dual}} \cdot d_p^k, \quad s^{k+1} := s^k + \beta_{\text{dual}} \cdot d_s^k$$

6. $k := k+1$, go to 2.

Complexity: $O(\sqrt{n} \log \frac{\epsilon_0}{\epsilon})$ iterations,
 (even $O(\log n \log \frac{\epsilon_0}{\epsilon})$ on average?)

Infeasible primal-dual path following method:

Start with $x^0 > 0$, $s^0 > 0$, p^0 that do not necessarily have to be feasible.

Only change: instead of the system (*) solve:

$$\begin{pmatrix} A & 0 & 0 \\ 0 & A^T & I \\ S_k & 0 & X_k \end{pmatrix} \cdot \begin{pmatrix} d_x^k \\ d_p^k \\ d_s^k \end{pmatrix} = - \begin{pmatrix} Ax^k - b \\ A^T p^k + s^k - c \\ X_k S_k e - \mu^k e \end{pmatrix}.$$

Summary:

Interior point methods start with an interior point of the feasible set and try to reach a primal/dual feasible solution pair with duality gap $< \epsilon$ by proceeding in a certain direction d_{method} in every step.

$$d_{\text{affine scaling}} = -X^2 (I - A^T (AX^2 A^T)^{-1} AX^2) c$$

$$d_{\text{path following}} = (I - A^T (AX^2 A^T)^{-1} AX^2) (Xe - \frac{1}{\mu} X^2 c)$$

$$\text{Let } d_{\text{center}} := (I - A^T (AX^2 A^T)^{-1} AX^2) Xe, \text{ then}$$

$$d_{\text{path following}} = d_{\text{center}} + \frac{1}{\mu} d_{\text{affine scaling}}$$

$$d_{\text{potential reduction}} = d_{\text{center}} + \frac{q}{s_{\text{TK}}} d_{\text{affine scaling}}$$

► Comparison with the simplex algorithm:

- in practice interior point methods tend to have a better performance for:
 - very large problems
 - massively degenerate problems
- interior point methods always find optimal solutions in the interior of the set of all optimal solutions (not a basic optimal solution)
 - IP
 - sensitive for unbounded sets of optimal solutions
 - problems with solving many similar problems