

Chapter 5': Integer Programming

(see chapter 5 of the book by Kortik & Vygen
"Combinatorial Optimization: Theory and Algorithms")

Given: $A \in \mathbb{Z}^{m \times n}$, $b \in \mathbb{Z}^m$, $c \in \mathbb{Z}^n$

Task: Find $x \in \mathbb{Z}^n$ such that $Ax \leq b$ and $c^T x$ is maximized.

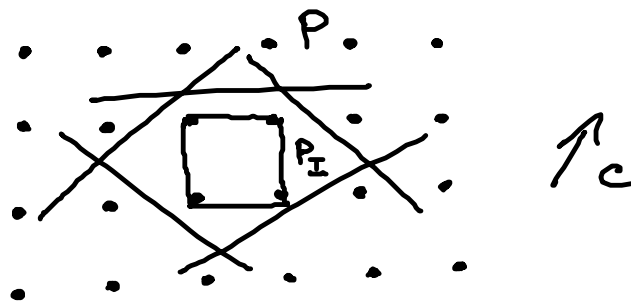
$$\begin{aligned} \max \quad & c^T x \\ \text{s.t.} \quad & Ax \leq b \\ & x \in \mathbb{Z}^n \end{aligned}$$

Integer Linear Program
(ILP)

(More general problem: Mixed integer linear program)
Only some variables are supposed to be integral.

Set of feasible solutions: $\{x \mid Ax \leq b, x \in \mathbb{Z}^n\}$

Note that $P = \{x \mid Ax \leq b\}$ is a polyhedron



Let $P_I = \{x \mid Ax \leq b\}_I$ the convex hull of the integral vectors in P ("integer hull of P ").

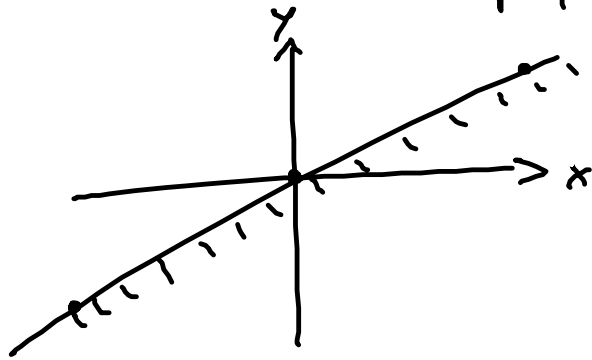
$$\{x \mid Ax \leq b, x \in \mathbb{Z}^n\} \subseteq P_I \subseteq P.$$

Remarks:

- (i) P bounded $\Rightarrow \{x \mid Ax \leq \delta, x \in \mathbb{Z}^n\}$ finite $\Rightarrow P_{\mathbb{I}}$ polyhedron
- (ii) P unbounded, A, δ rational $\Rightarrow P_{\mathbb{I}}$ polyhedron (see later)
- (iii) P unbounded, A, δ arbitrary (real), then $P_{\mathbb{I}}$ is in general not a polyhedron.

Example: $P = \{(x, y) \in \mathbb{R}^2 \mid y \leq \sqrt{2} \cdot x\}$

Exercise: $P_{\mathbb{I}}$ is not a polyhedron.



Important question: Under which conditions is $P_{\mathbb{I}} = P$? In this case the ILP can be solved by solving the LP relaxation:

$$\begin{aligned} \max & \quad c^T x \\ \text{s.t.} & \quad Ax \leq \delta \\ & \quad (x \in \mathbb{Z}^n) \end{aligned}$$

A more general task: Given polyhedron P , characterize the integer hull $P_{\mathbb{I}}$ of P .

5.1: The integer hull of a polyhedron

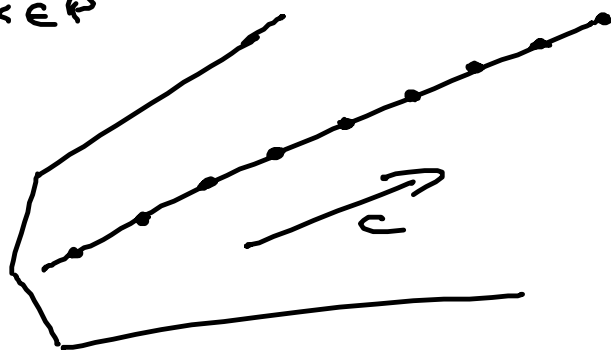
An ILP is either infeasible, has an opt. solution or is unbounded. Infeasibility is difficult to detect. But:

Proposition: Let $P = \{x \mid Ax \leq \delta\}$ with A, δ rational and $P_{\mathbb{I}} \neq \emptyset$. For $c \in \mathbb{Q}^n$

max $c^T x$
s.t. $x \in P$

unbounded \Leftrightarrow

max $c^T x$
s.t. $x \in P_I$ unbounded



Proof: " \Leftarrow " clear

" \Rightarrow " Suppose that max $c^T x$ s.t. $x \in P$ unbounded

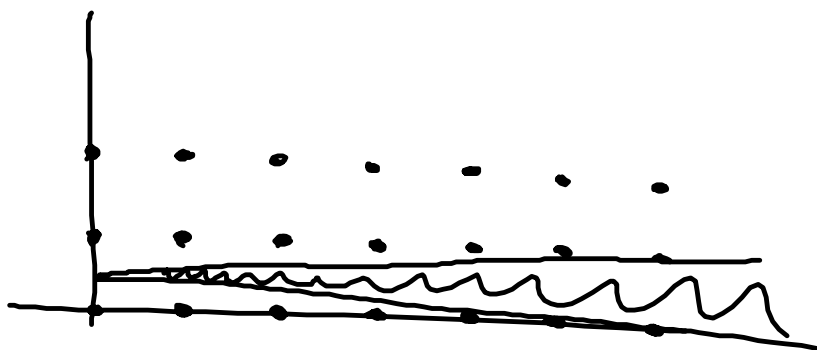
$$\Rightarrow \exists z \in \mathbb{R}^n \text{ with } c^T z > 0 \text{ and } A \cdot z \leq 0$$

$$\Rightarrow \exists z \in \mathbb{Q}^n \text{ with } c^T z = 1 \text{ and } A \cdot z \leq 0$$

$$\Rightarrow \exists z \in \mathbb{Z}^n \text{ with } c^T z > 0 \text{ and } A \cdot z \leq 0$$

Let $y \in \mathbb{Z}^n$ with $Ay \leq b \Rightarrow y + k \cdot z \in P_I \cap \mathbb{Z}^n \forall k \in \mathbb{N}$

$$\text{and } c^T (y + k \cdot z) \xrightarrow{k \rightarrow \infty} \infty. \quad \square$$

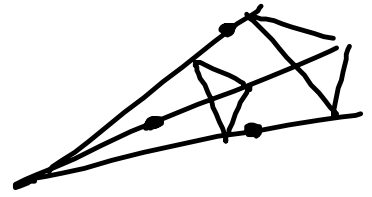


Def: For $A \in \mathbb{Z}^{m \times n}$, a subdeterminant of A is $\det B$ for some square submatrix B of A .
Let $\Theta(A)$ be the maximum absolute value of the subdeterminants of A .

Lemma: Let $A \in \mathbb{Z}^{m \times n}$ and $C = \{x \mid Ax \geq 0\}$ a polyhedral cone. Then C is generated

by a finite set of integral vectors γ_i , each having components with absolute value at most $\Theta(A)$, i.e.,

$$C = \left\{ \sum_i \lambda_i \gamma_i \mid \lambda_i \geq 0 \forall i \right\}$$
$$\|\gamma_i\|_\infty \leq \Theta(A).$$



Proof: see book. \square