Chapter 5': Integer Programming

(see chapter 5 of the book by Korte & Vygen "Combinatorial Optimization: Theory and Algorithms")

<u>Given</u>: $A \in \mathbb{Z}^{m \times n}$, $b \in \mathbb{Z}^m$, $c \in \mathbb{Z}^n$

<u>Task</u>: Find $x \in \mathbb{Z}^n$ such that $Ax \leq b$ and $c^T x$ is maximized.

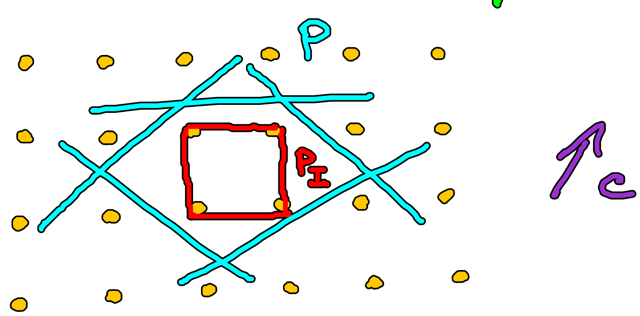
$$\begin{aligned} \max\ & c^T x \\ \text{s.t. } & Ax \leq b \\ & x \in \mathbb{Z}^n \end{aligned}$$

Integer Linear Program (ILP)

(<u>More general problem</u>: Mixed integer linear program. Only some variables are supposed to be integral.)

Set of feasible solutions: $\{x \mid Ax \leq b, x \in \mathbb{Z}^n\}$

Note that $P = \{x \mid Ax \leq b\}$ is a polyhedron



Let $P_I = \{x \mid Ax \leq b\}_I$ the convex hull of the integral vectors in P ("integer hull of P").

$$\{x \mid Ax \leq b, x \in \mathbb{Z}^n\} \subseteq P_I \subseteq P.$$

<u>Remarks:</u>

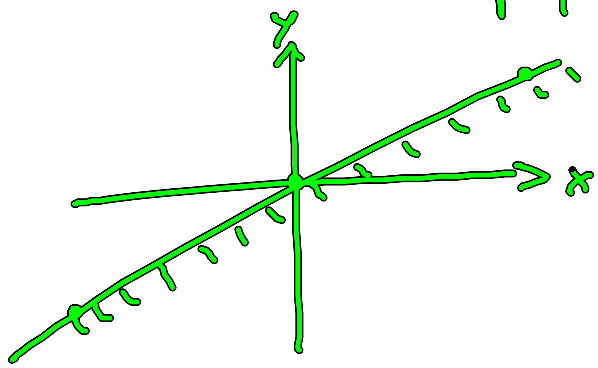
(i) P bounded $\Rightarrow \{x \mid Ax \leq b, x \in \mathbb{Z}^n\}$ finite $\Rightarrow P_I$ polyhedron

(ii) P unbounded, A, b rational $\Rightarrow P_I$ polyhedron (see later)

(iii) P unbounded, A, b arbitrary (real), then P_I is in general not a polyhedron.

Example: $P = \{(x,y) \in \mathbb{R}^2 \mid y \leq \sqrt{2} \cdot x\}$

Exercise: P_I is not a polyhedron.



<u>Important question:</u> Under which conditions is $P_I = P$? In this case the ILP can be solved by solving the LP relaxation:

$$\begin{aligned} \max\ & c^T x \\ \text{s.t.}\ & Ax \leq b \\ & (x \in \mathbb{Z}^n) \end{aligned}$$

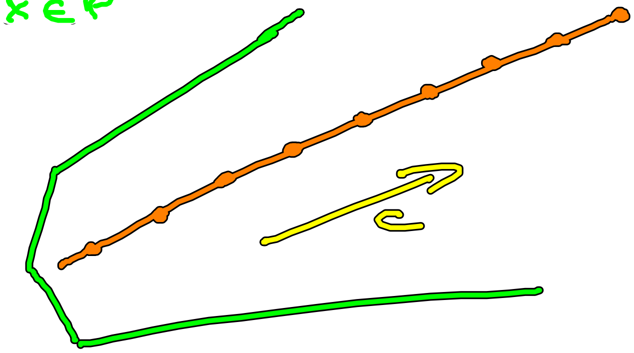
A more general task: Given polyhedron P, characterize the integer hull P_I of P.

<u>5.1: The integer hull of a polyhedron</u>

An ILP is either infeasible, has an opt. solution or is unbounded. Infeasibility is difficult to detect. But:

<u>Proposition:</u> Let $P = \{x \mid Ax \leq b\}$ with A, b rational and $P_I \neq \emptyset$. For $c \in \mathbb{Q}^n$

$\max c^T x$ s.t. $x \in P$ unbounded $\Longleftrightarrow$ $\max c^T x$ s.t. $x \in P_I$ unbounded



Proof: "$\Leftarrow$" clear

"$\Rightarrow$" Suppose that $\max c^T x$ s.t. $x \in P$ unbounded

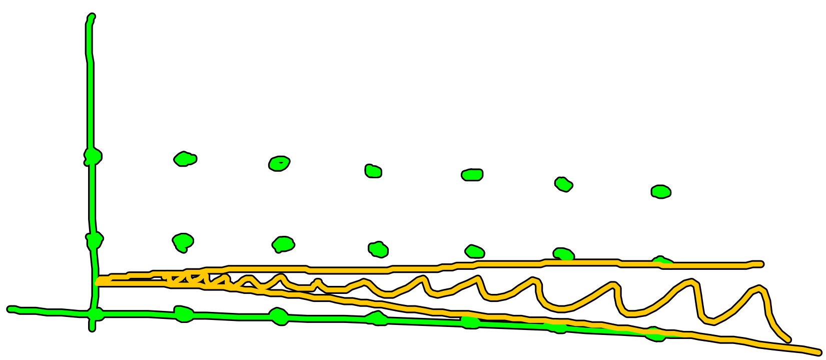
$\Rightarrow \exists\, z \in \mathbb{R}^n$ with $c^T z > 0$ and $A \cdot z \leq 0$

$\Rightarrow \exists\, z \in \mathbb{Q}^n$ with $c^T z = 1$ and $A \cdot z \leq 0$

$\Rightarrow \exists\, z \in \mathbb{Z}^n$ with $c^T z > 0$ and $A \cdot z \leq 0$

Let $y \in \mathbb{Z}^n$ with $Ay \leq b \Rightarrow y + k \cdot z \in P_I \cap \mathbb{Z}^n \;\forall k \in \mathbb{N}$

and $c^T(y + k \cdot z) \xrightarrow[k \to \infty]{} \infty$. $\square$

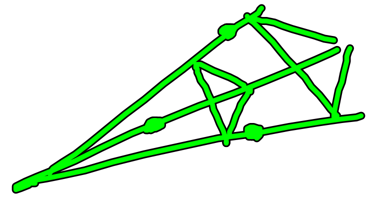


Def: For $A \in \mathbb{Z}^{m \times n}$, a subdeterminant of A is $\det B$ for some square submatrix B of A. Let $\Theta(A)$ be the maximum absolute value of the subdeterminants of A.

Lemma: Let $A \in \mathbb{Z}^{m \times n}$ and $C = \{x \mid Ax \geq 0\}$ a polyhedral cone. Then C is generated by a finite set of integral vectors y_i, each having components with absolute value at most $\Theta(A)$, i.e.,

$$C = \left\{ \sum_i \lambda_i y_i \;\middle|\; \lambda_i \geq 0 \;\forall i \right\}$$

$$\|y_i\|_\infty \leq \Theta(A).$$



Proof: see book. $\square$