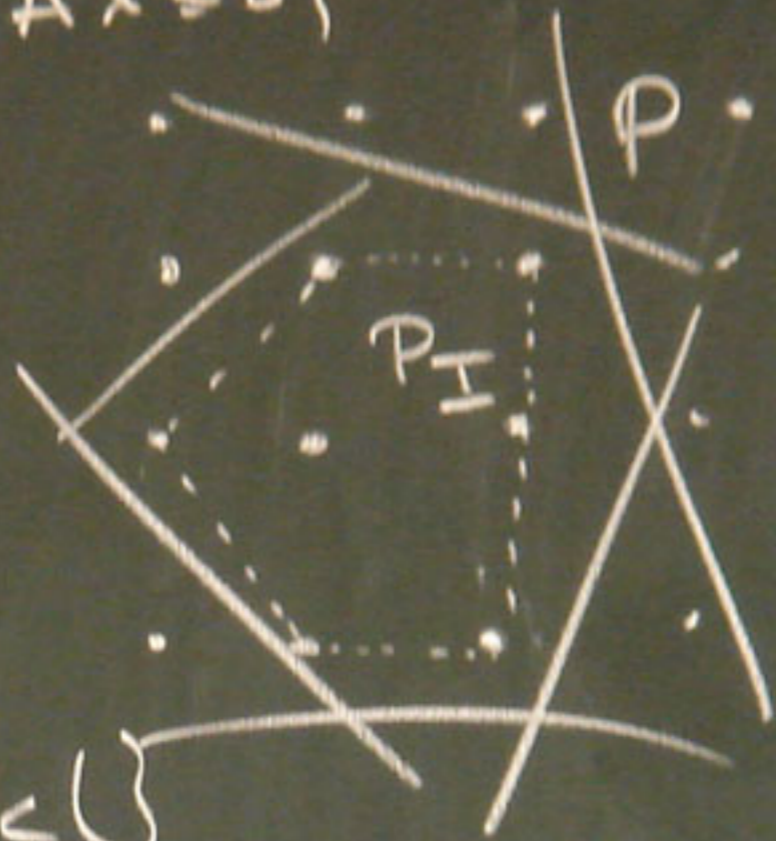


$$\max c^T x \quad P = \{x \mid Ax \leq b\}$$

$$\text{s.t. } Ax \leq b$$

x integer

$$x \in \mathbb{Z}^n$$

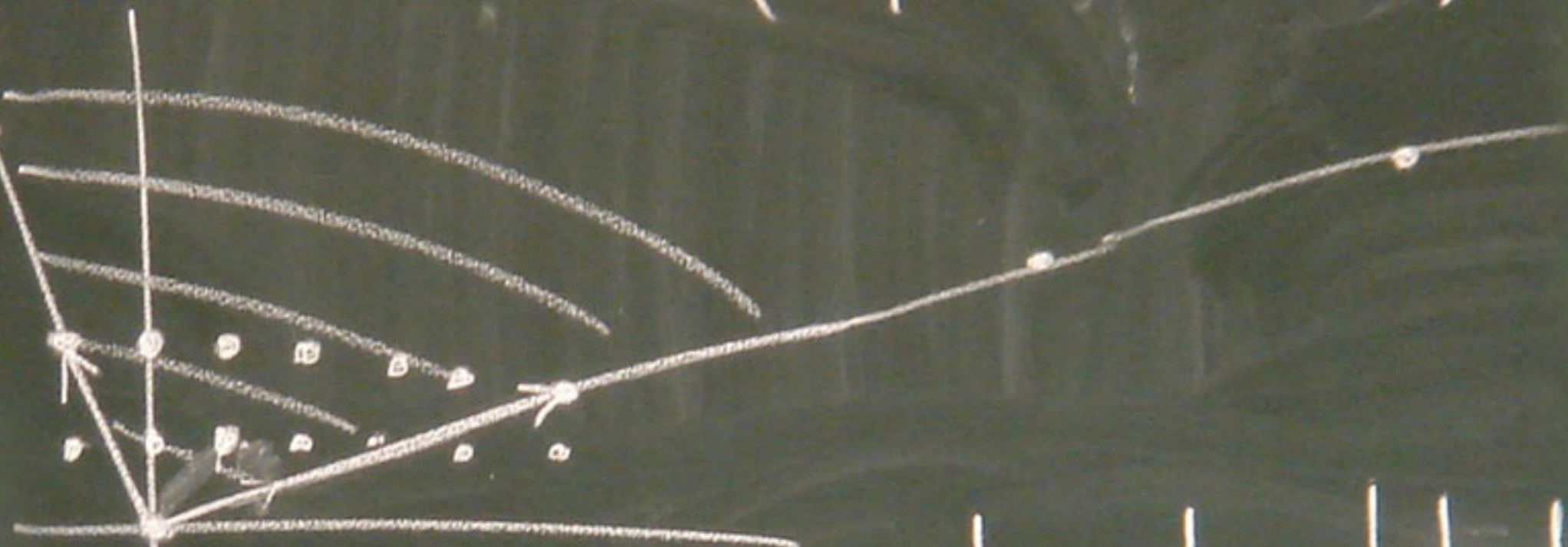


$$P_I = \text{conv} \{x \in \mathbb{Z}^n \mid Ax \leq b\}$$

Lemma. $A \in \mathbb{Z}^{m \times n}$ and $C = \{x \mid Ax \geq 0\}$

C is generated by a finite set of integral vectors y_i each having components

with absolute value at
most $\Theta(A)$. ($\|y_i\|_\infty \leq \Theta(A)$)



Lemma 5.4. Each rational polyhedral
cone C is generated by a finite set
of integral vectors $\{a_1, \dots, a_t\}$ such
that each integral vector x in C is

a non-negative integral comb
of a_1, \dots, a_t , i.e.

$$x = \sum_{i=1}^t \lambda_i \cdot a_i \quad \lambda_i \geq 0$$

("Hilbert basis" of C)
 $\lambda_i \in \mathbb{Z}$

Proof: C is generated by
integral vectors b_1, \dots, b_k .
Let a_1, \dots, a_t be all integral
vectors in the polytope

$$P = \{ \lambda_1 b_1 + \dots + \lambda_k b_k \mid 0 \leq \lambda_i \leq 1 \text{ for } i=1, \dots, k \}$$

Since $\{b_1, \dots, b_k\} \subseteq \{a_1, \dots, a_k\}$,
 the a_i 's generate C .

Let $x \in C \cap \mathbb{R}^n$

$$x = \sum_{i=1}^k \mu_i b_i, \quad \mu_i \geq 0$$

$$= \sum_{i=1}^k L[\mu_i] \cdot b_i +$$

$$\underbrace{\sum_{i=1}^k (\mu_i - L[\mu_i]) \cdot b_i}_{\in P \cap \mathbb{R}^n}$$

$$\in P \cap \mathbb{R}^n \\ = a_j$$

Theorem: $A \in \mathbb{Z}^{m \times n}$, $b \in \mathbb{Z}^m$, $c \in \mathbb{R}^n$,

$P = \{x \mid Ax \leq b\}$ and $P_{\mathbb{I}} \neq \emptyset$.

(a) If y is opt. sol. of $\max c^T x$ s.t. $x \in P$

then there is an opt. integral sol z
of $\max c^T x$ s.t. $x \in P_{\mathbb{I}}$ with

$$\|z - y\|_{\infty} \leq n \cdot \Theta(A)$$

(b) If y is a feasible integral but not opt.

solution to $\max c^T x$ s.t. $x \in P_{\mathbb{I}}$, then there

is $z \in P_{\mathbb{I}} \cap \mathbb{Z}^n$ with $c^T z > c^T y$ and $\|z - y\|_{\infty} \leq n \cdot \Theta(A)$

Proof. Let $y \in P$ arbitrary and z^* opt. integral solution of $\max c^T x$ s.t. $x \in P_{\mathbb{I}}$. Split $Ax \leq b$ into two subsystems

$$A_1 x \leq b_1$$

$$A_2 x \leq b_2$$

with

$$A_1 z^* \geq A_1 y$$

$$A_2 z^* < A_2 y$$

$$\Rightarrow \underline{z^* - y} \in C = \{x \mid A_1 x \geq 0, A_2 x \leq 0\}$$

C is generated by $x_1, \dots, x_s \in \mathbb{Z}^n$ with $\|x_i\|_{\infty} \leq \theta(A)$

$$\Rightarrow z^* - y = \sum_{i=1}^s \lambda_i x_i \quad \text{for } \lambda_i \geq 0$$

with at most n of the λ_i 's
positive (Carathéodory).

For $\mu = (\mu_1, \dots, \mu_s)$ with $0 \leq \mu_i \leq \lambda_i$

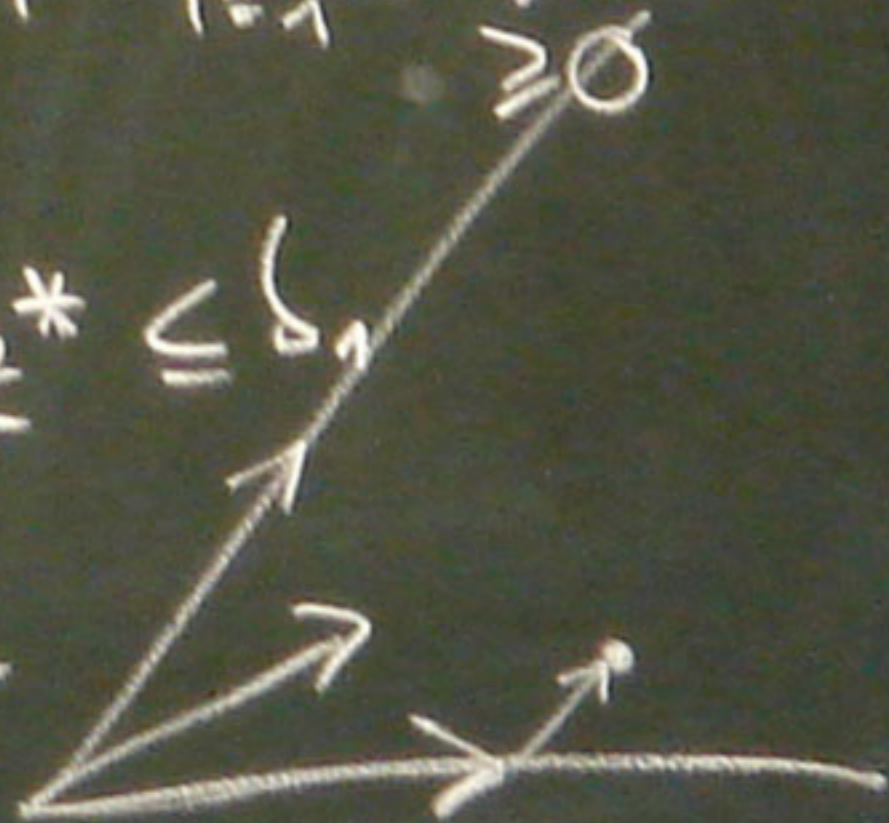
define

$$z_\mu := z^* - \underbrace{\sum_{i=1}^s \mu_i x_i}_{\geq 0} = y + \underbrace{\sum_{i=1}^s (x_i - \mu_i) x_i}_{\geq 0}$$

Then: $A_1 z_\mu \leq A_1 z^* \leq b_1$

$A_2 z_\mu \leq A_2 y \leq b_2$

$\Rightarrow z_\mu \in P.$



Case 1: $\exists i \in \{1, \dots, s\} : \lambda_i \geq 1$ and $c^T x_i > 0$.

Let $z := y + x_i \Rightarrow z \in P$ and $c^T z > c^T y$ (\downarrow in (a))

For (b): y integral $\Rightarrow z$ integral and $\|z - y\|_\infty = \|x_i\|_\infty \leq \theta(A)$.

Case 2: $\forall i \in \{1, \dots, s\} : \lambda_i \geq 1 \Rightarrow c^T x_i \leq 0$

Let $z := z_{\lfloor \lambda \rfloor} = z^* - \sum_{i=1}^s \lfloor \lambda_i \rfloor x_i \in \mathcal{P} \cap \mathbb{Z}^n$

and $c^T z \geq c^T z^*$ ($\Rightarrow z$ opt. ILP solution)

and $\|z - y\|_\infty = \left\| \sum_{i=1}^s (\lambda_i - \lfloor \lambda_i \rfloor) x_i \right\|_\infty \leq \sum_{i=1}^s \underbrace{(\lambda_i - \lfloor \lambda_i \rfloor)}_{\leq 1} \cdot \underbrace{\|x_i\|_\infty}_{\leq \Theta(A)} \leq n \cdot \Theta(A) \quad \square$

Corollary: An opt sol to an ILP
can be encoded polynomially
in the input size.

\Rightarrow Integer linear inequalities is in NP.

By part (b) of the theorem above, optimality
of $y \in P_{\mathbb{I}^n} \subseteq \mathbb{Z}^n$ can be checked by testing
 $y+x$ for a finite set of vectors x that depend

only on A . ("test set").

Test sets are sometimes useful in practice.

Theorem: For each $A \in \mathbb{Z}^{m \times n}$ there exists $M \in \mathbb{Z}^{m' \times n}$

(with $|M_{ij}| \leq n^{2n} \Theta(A)^n$) such that for each

$\delta \in \mathbb{Q}^m$ there is $d \in \mathbb{Q}^{m'}$ with:

$$P_{\mathbb{I}} = \{x \mid Ax \leq \delta\}_{\mathbb{I}} = \{x \mid Mx \leq d\}$$