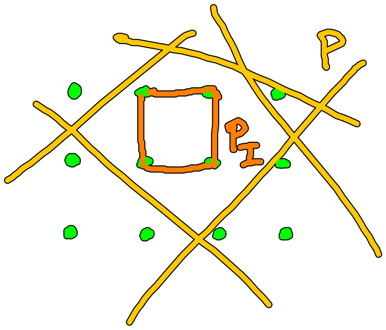


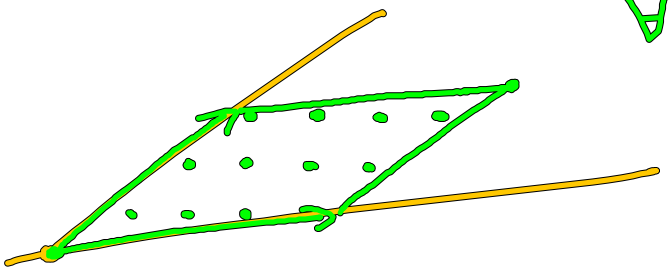
$C = \{x \mid Ax \geq 0\}$ $\quad A \in \mathbb{Q}^{m \times n}$

$\longrightarrow \exists$ Hilbert basis $\{a_1, \dots, a_t\}$:

$C = \text{cone}(\{a_1, \dots, a_t\})$ and

$\forall\, x \in C \cap \mathbb{Z}^n$:

$$x = \sum_{i=1}^{t} \lambda_i \cdot a_i \text{ with } \lambda_i \geq 0,\ \lambda_i \in \mathbb{Z}$$



---

x opt. LP solution

y opt ILP solution $\quad \|x - y\|_\infty \leq n \cdot \Theta(A)$

$P = \{x \mid Ax \leq b\}$ $\quad \max c^T x$ s.t. $x \in P \cap \mathbb{Z}^n$

y_1 feasible ILP solution, not optimal

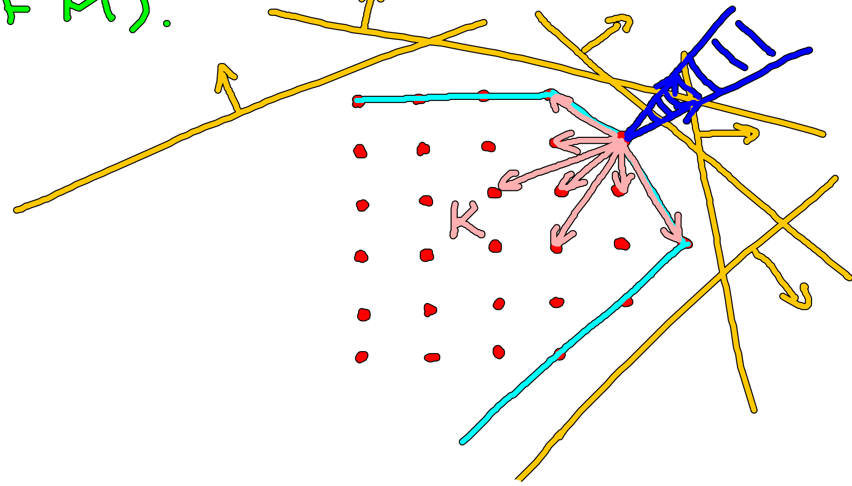
$\longrightarrow \exists\, y_2$ feasible ILP solution with $c^T y_2 > c^T y_1$

and $\|y_1 - y_2\|_\infty \leq n \cdot \Theta(A)$

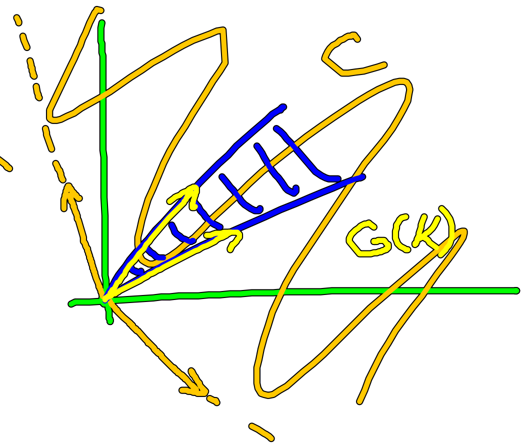
Theorem: For each $A \in \mathbb{Z}^{m \times n}$ there exists $M \in \mathbb{Z}^{m' \times n}$ (with $|M_{ij}| \leq n^{2n} \cdot \Theta(A)^n$) such that for each $b \in \mathbb{Q}^m$ there is $d \in \mathbb{Q}^{m'}$ with

$$P_I = \{x \mid Ax \leq b\}_I = \{x \mid M \cdot x \leq d\}.$$

Proof: (we don't prove the bound on the entries of M).



Let $C:=\{A^T\cdot\mu \mid \mu\geq 0\}$

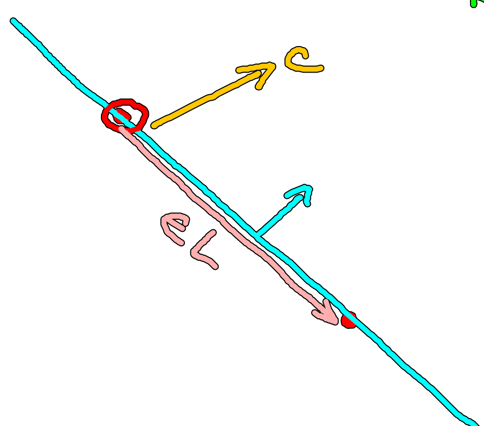


$L:=\{z\in\mathbb{Z}^n \mid \|z\|_\infty \leq n\cdot\Theta(A)\}$ "test set"

For $K\subseteq L$ let $C_K := C\cap \boxed{\{y \mid z^T\cdot y\leq 0\ \forall z\in K\}}$

C_K is a polyhedral cone (note that C is finitely generated) and therefore generated by a finite set of integer vectors $G(K)\subseteq\mathbb{Z}^n$.

Let $M\in\mathbb{Z}^{m'\times n}$ with rows $\bigcup_{K\subseteq L} G(K)$.



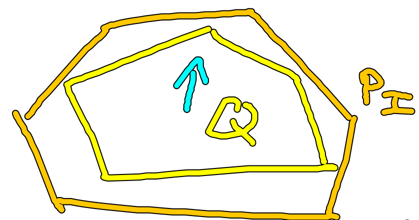
Assume that $P_I\neq\emptyset$ (otherwise see book).

and let $\delta_y := \max\{y^T \cdot x \mid Ax \le b, x \in \mathbb{Z}^n\}$ for all $y \in \bigcup_{K \in L} G(K)$

We will show that $P_I = \{x \mid y^T x \le \delta_y \ \forall y \in \bigcup_{K \in L} G(K)\}$
$= \underbrace{\{x \mid M \cdot x \le \delta\}}_{Q :=}$

$P_I \subseteq Q$: clear by definition of δ_y.

$P_I \supseteq Q$:



Show that
$\max\{c^T x \mid x \in P_I\} \ge \max\{c^T x \mid x \in Q\} \ \forall c \in \mathbb{R}^n$

Consider c with $\max\{c^T x \mid x \in P_I\} < \infty$, $x^* \in P_I \cap \mathbb{Z}^n$ maximizes $c^T x$

$\Rightarrow \max\{c^T x \mid Ax \le b\} < \infty$
$\Rightarrow \min\{y^T \cdot b \mid y^T A = c^T, y \ge 0\}$ feasible
$\Rightarrow c \in C$

Let $\overline{K} := \{z \in L \mid A \cdot (x^* + z) \le b\}$

Since x^* is optimal $\Rightarrow c^T \cdot z \le 0 \ \forall z \in \overline{K} \Rightarrow c \in C_{\overline{K}}$

$\Rightarrow c = \sum_{y \in G(\overline{K})} \lambda_y \cdot y \qquad \lambda_y \ge 0$

<u>Claim:</u> $\delta_y := \max\{y^T \cdot x \mid x \in P_I\} = y^T x^* \ \forall y \in G(\overline{K})$

<u>Proof:</u> Otherwise $\exists z \in \overline{K}$ with $y^T z > 0$ ↯ $y \in C_{\overline{K}}$.

$\Rightarrow$ For $x \in Q$:

$$c^T x = \sum_{y \in G(\overline{K})} (\lambda_y \cdot y^T) \cdot x$$

$$\leq \sum_{y \in G(\bar{K})} \lambda_y \cdot \delta_y$$

$$= \sum_{y \in G(\bar{K})} (\lambda_y \cdot y^T) x^* = c^T \cdot x^* \quad \square$$

Total dual integrality

Def: A polyhedron P is integral if $P = P_I$.

Some definitions and notions on polyhedra

Let $P = \{x \mid Ax \leq b\}$ polyhedron

If $c \neq 0$, $c^T x \leq \delta \ \forall x \in P$ ($c^T x \leq \delta$ is a "valid inequality" for P)

and $P \cap \{x \mid c^T x = \delta\} \neq \emptyset$ then

$\{x \mid c^T x = \delta\}$ is called a supporting hyperplane of P.

A face of P is P itself or the intersection of P with a supporting hyperplane.

A face of a polyhedron is itself a polyhedron.

A facet of P is a maximal face distinct from P.

Theorem: Let P be a rational polyhedron. The following statements are equivalent

a) P is integral

b) Each face of P contains integral vectors.

c) Each minimal face of P contains integral vectors.

d) Each supporting hyperplane contains integral vectors

e) Each rational support. hyperpl. " "
vectors

f) $\max\{c^T x \mid x \in P\} = \max\{c^T x \mid x \in P \cap \mathbb{Z}^n\} \; \forall c \in \mathbb{R}^n$

g) $\max\{c^T x \mid x \in P\} \in \mathbb{Z} \cup \{\infty\} \; \forall c \in \mathbb{Z}^n$