

Total dual integrality

P polyhedron

P integral if $P = P_I$

Theorem: The following are equivalent:

- P integral
- Each face of P contains integral vectors
- Each min. face " " " "
- Each supp. hyperpl. " " " "
- Each rational supp. hyperpl. " " " "
- $\max \{c^T x \mid x \in P\} = \max \{c^T x \mid x \in P \cap \mathbb{Z}^n\} \quad \forall c \in \mathbb{R}^n$
- $\max \{c^T x \mid x \in P\} \in \mathbb{Z} \cup \{\infty\} \quad \forall c \in \mathbb{Z}^n$

Def.: $Ax \leq b$ is called totally dual integral (TDI) if for each $c \in \mathbb{Z}^n$ with $\max \{c^T x \mid Ax \leq b\} < \infty$:
 $\max \{c^T x \mid Ax \leq b\} = \min \{y^T b \mid y^T A = c^T, y \geq 0, y \in \mathbb{Z}^m\}$

Corollary: Let $A \in \mathbb{Q}^{m \times n}$, $b \in \mathbb{Z}^m$ and $Ax \leq b$ TDI.
Then $\{x \mid Ax \leq b\}$ is integral.

Proof: Follows from the definition of TDI and "g) \Rightarrow a)" of the above theorem. \square

Application / Example:

Given: Directed graph $D = (V, A)$, $s, t \in V$

$\mathcal{P} = \{s-t\text{-paths in } D\}$, $c: A \rightarrow \mathbb{R}_+$

Problem: Assign weights $y_a \geq 0, a \in A$, to the arcs such that the weight of any s-t-path is at least 1. Objective: minimize $\sum_{a \in A} c(a) \cdot y_a$

LP Formulation:

$$\min \sum_{a \in A} c(a) \cdot y_a$$

$$\text{s.t. } \sum_{a \in P} \gamma_a \geq 1 \quad \forall P \in \mathcal{P}$$

$$\gamma_a \geq 0$$

Claim: This polyhedron $Q = \{ \gamma \mid \sum_{a \in P} \gamma_a \geq 1 \quad \forall P \in \mathcal{P}, \gamma \geq 0 \}$ is integral because it is TDI.

$$\max \sum_{P \in \mathcal{P}} 1 \cdot x_P$$

$$\text{s.t. } \sum_{P: a \in P} x_P \leq c(a) \quad \forall a \in A$$

$$x_P \geq 0 \quad \forall P \in \mathcal{P}$$

This is a formulation of the max s-t-flow problem. If the "capacities" $c(a)$ are integral $\forall a \in A$, then there exists an integral maxflow.

The corollary above thus implies that Q is integral.

Notice: TDI-ness is not a property of polyhedra but a property of linear inequality systems. There can be two different descriptions of a particular polyhedron by inequality systems such that one is TDI and the other one is not TDI.

Proposition: If $Ax \leq b$ is TDI and $a^T x \leq \beta$ for all x with $Ax \leq b$, then $Ax \leq b, a^T x \leq \beta$ is also TDI.

Proof: For $c \in \mathbb{Z}^n$

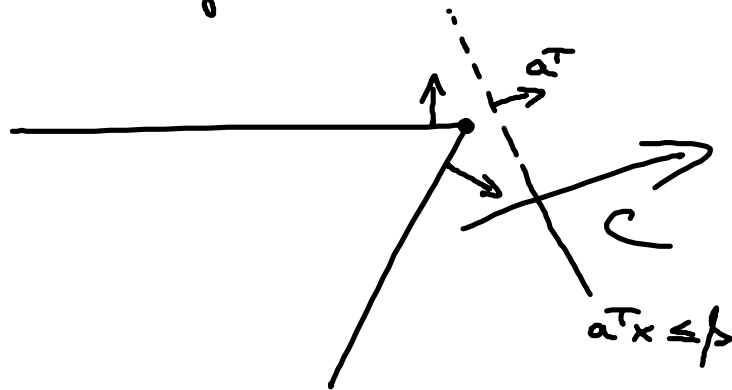
$$\min \{ y^T b \mid y^T A = c^T, y \geq 0 \} = \max \{ c^T x \mid Ax \leq b \}$$

(*)

$$= \max \{ c^T x \mid Ax \leq b, a^T x \leq \beta \}$$

$$= \min \{ \gamma^T b + \gamma \cdot \beta \mid \gamma^T A + \gamma \cdot a^T = c^T, \gamma, \gamma \geq 0 \}$$

Since $Ax \leq b$ is TPI, $(*)$ has an integral opt. sol. γ . Then, $(\gamma, 0)$ is an integral opt. sol. to $(**)$. □

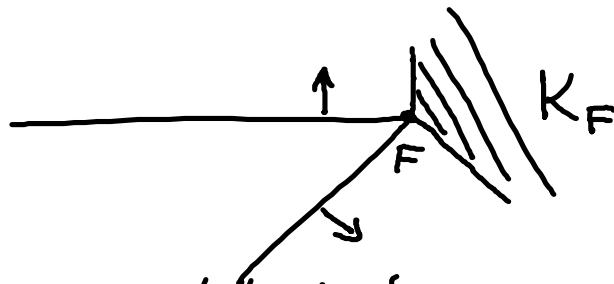


Theorem: For each rational polyhedron P there exists $Ax \leq b$ TPI with A integral and $P = \{ x \mid Ax \leq b \}$. In particular $b \in \mathbb{Z}^m \Leftrightarrow P$ integral.

Sketch of the proof:

Let $P = \{ x \mid Cx \leq d \}$, $C \in \mathbb{Q}^{m \times n}$, $d \in \mathbb{Q}^m$ and F minimal face of P , $F = \{ x \mid C'x = d' \}$ with $C'x \leq d'$ a subsystem of $Cx \leq d$.

$$K_F := \{ c \mid c^T z = \max \{ c^T x \mid x \in P \} \forall z \in F \}$$



The K_F is a polyhedral cone generated by the rows of C' . (omit details)

Let a_1, \dots, a_t be a Hilbert basis of K_F and

Let $\max \{a_i^T x \mid x \in P\} = b_i$

Consider the resulting system $Ax \leq b$ (for all F).

(If P integral, then b integral)

$$\rightarrow P = \{x \mid Ax \leq b\}$$

"1" by Def. of r.h.s. b_i .

"2" Positive multiples of the rows of C must occur as rows of A since the rows of C^i generate the polyh. cone K_F .

It remains to prove that $Ax \leq b$ is T.D.I.

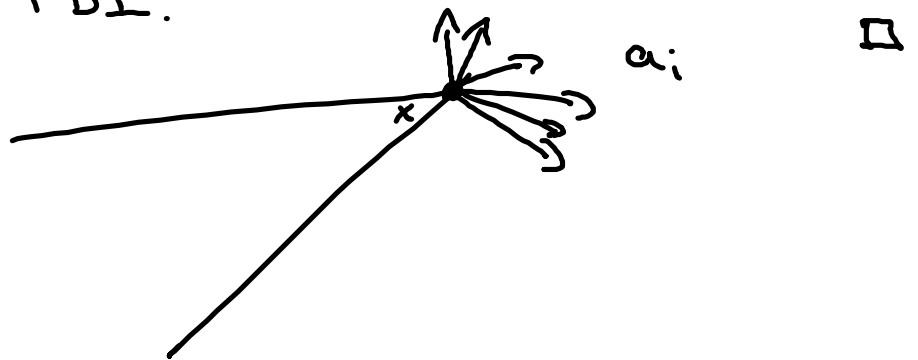
Let $c \in \mathbb{Z}^n$ and F minimal face of P with $\max \{c^T x \mid x \in P\} = c^T z \quad \forall z \in F$

$$\rightarrow c \in K_F, \quad c = \sum_{i=1}^t \lambda_i a_i, \quad \lambda_i \in \mathbb{Z}_+$$

$$\Rightarrow c^T = \lambda^T A \quad \text{for some } \lambda \in \mathbb{Z}_+^{m'}$$

$$\Rightarrow \lambda^T \cdot b = \lambda^T \cdot (Ax) = (\lambda^T A) \cdot x = c^T \cdot x \quad \forall x \in F$$

$$\Rightarrow Ax \leq b \text{ is T.D.I.}$$



Theorem: Let $Ax \leq b, a^T x = \beta$ T.D.I. and a^T integral $\rightarrow Ax \leq b, a^T x = \beta$ T.D.I.