Total dual integrality

P polyhedron

P integral if $P = P_I$

Theorem: The following are equivalent:

a) P integral

b) Each face of P contains integral vectors

c) Each min. face " " " "

d) Each supp. hyperpl. " " " "

e) Each rational supp. hyperpl. " " "

f) $\max\{c^Tx \mid x \in P\} = \max\{c^Tx \mid x \in P \cap \mathbb{Z}^n\}$ $\forall c \in \mathbb{R}^n$

g) $\max\{c^Tx \mid x \in P\} \in \mathbb{Z} \cup \{\infty\}$ $\forall c \in \mathbb{Z}^n$

Def.: $Ax \le b$ is called totally dual integral (TDI) if for each $c \in \mathbb{Z}^n$ with $\max\{c^Tx \mid Ax \le b\} < \infty$:

$$\max\{c^Tx \mid Ax \le b\} = \min\{y^Tb \mid y^TA = c^T, y \ge 0, y \in \mathbb{Z}^m\}$$

Corollary: Let $A \in \mathbb{Q}^{m \times n}$, $b \in \mathbb{Z}^m$ and $Ax \le b$ TDI. Then $\{x \mid Ax \le b\}$ is integral.

Proof: Follows from the definition of TDI and "g) => a)" of the above theorem. □

Application / Example:

Given: Directed graph $D = (V, A)$, $s, t \in V$

$\mathcal{P} = \{s\text{-}t\text{-paths in } D\}$, $c: A \to \mathbb{R}_+$

Problem: Assign weights $y_a \ge 0$, $a \in A$, to the arcs such that the weight of any s-t-path is at least 1. Objective: minimize $\sum_{a \in A} c(a) \cdot y_a$

LP formulation:

$$\min \sum_{a \in A} c(a) \cdot y_a$$

$$\text{s.t.} \sum_{a \in P} y_a \geq 1 \quad \forall P \in \mathcal{P}$$
$$y_a \geq 0$$

<u>Claim:</u> This polyhedron $Q = \{ y \mid \sum_{a \in P} y_a \geq 1 \ \forall P \in \mathcal{P}, y \geq 0 \}$ is integral because it is TDI.

$$\max \sum_{P \in \mathcal{P}} 1 \cdot x_P$$
$$\text{s.t.} \sum_{P : a \in P} x_P \leq c(a) \quad \forall a \in A$$
$$x_P \geq 0 \quad \forall P \in \mathcal{P}$$

This is a formulation of the max s-t-flow problem. If the "capacities" $c(a)$ are integral $\forall a \in A$, then there exists an integral maxflow.

The corollary above thus implies that Q is integral.

<u>Notice:</u> TDIness is not a property of polyhedra but a property of linear inequality systems. There can be two different descriptions of a particular polyhedron by inequality systems such that one is TDI and the other one is not TDI.

<u>Proposition:</u> If $Ax \leq b$ is TDI and $a^T x \leq \beta$ for all x with $Ax \leq b$, then $Ax \leq b$, $a^T x \leq \beta$ is also TDI.

<u>Proof:</u> For $c \in \mathbb{Z}^n$

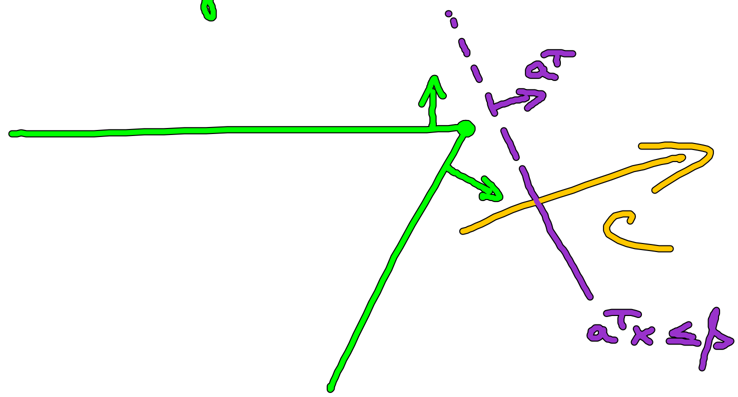
$$\min \{ y^T b \mid \underset{(*)}{y^T A = c^T}, y \geq 0 \} = \max \{ c^T x \mid Ax \leq b \}$$

$= \max\{c^T x \mid Ax \le b, a^T x \le \beta\}$

$= \min\{y^T b + \gamma \cdot \beta \mid \underbrace{y^T A + \gamma \cdot a^T = c^T, y, \gamma \ge 0}_{(**)}\}$

Since $Ax \le b$ is TDI, $(*)$ has an integral opt. sol. y. Then, $(y, \underset{\gamma}{\overset{\shortparallel}{0}})$ is an integral opt. sol. to $(**)$.

$\square$

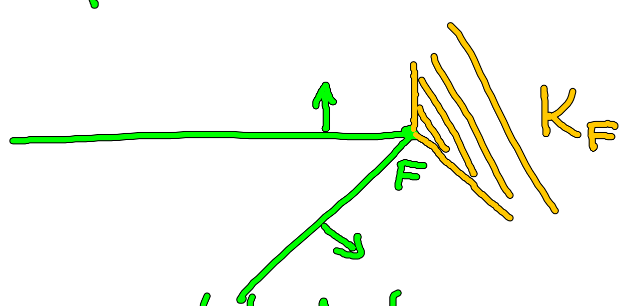


Theorem: For each rational polyhedron P there exists $Ax \le b$ TDI with A integral and $P = \{x \mid Ax \le b\}$. In particular $b \in \mathbb{Z}^m \Leftrightarrow P$ integral.

Sketch of the proof:

Let $P = \{x \mid Cx \le d\}$, $C \in \mathbb{Q}^{m \times n}$, $d \in \mathbb{Q}^m$ and F minimal face of P, $F = \{x \mid C'x = d'\}$ with $C'x \le d'$ a subsystem of $Cx \le d$.

$K_F := \{c \mid c^T z = \max\{c^T x \mid x \in P\} \ \forall z \in F\}$



Then K_F is a polyhedral cone generated by the rows of C'. (omit details)

Let $a_1, \ldots, a_t$ be a Hilbert basis of K_F and let $\max\{a_i^T x \mid x \in P\} = b_i$

Consider the resulting system $Ax \le b$ (for all F).

((If P integral, then b integral))

$\Rightarrow P = \{x \mid Ax \le b\}$

"$\le$" by Def. of r.h.s. b_i.

"$\ge$" Positive multiples of the rows of C must occur as rows of A since the rows of C' generate the polyh. cone K_F.

It remains to prove that $Ax \le b$ is TDI.

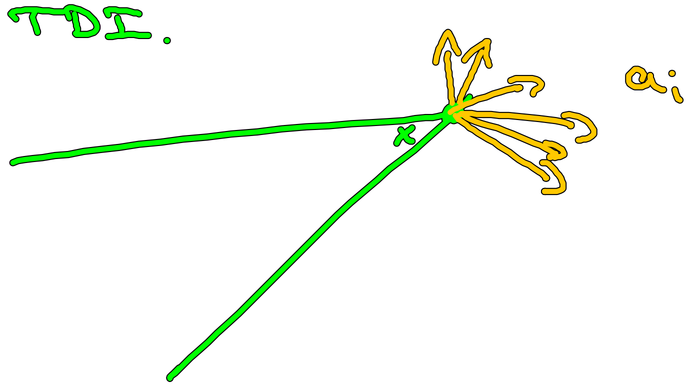
Let $c \in \mathbb{Z}^n$ and F minimal face of P with $\max\{c^T x \mid x \in P\} = c^T z \;\forall z \in F$

$\Rightarrow c \in K_F$, $c = \sum_{i=1}^{t} \lambda_i a_i$, $\lambda_i \in \mathbb{Z}_+$

$\Rightarrow c^T = \lambda^T A$ for some $\lambda \in \mathbb{Z}_+^{m'}$

$\Rightarrow \lambda^T \cdot b = \lambda^T \cdot (Ax) = (\lambda^T A) \cdot x = c^T \cdot x \quad \forall x \in F$

$\Rightarrow Ax \le b$ is TDI. $\square$



<u>Theorem:</u> Let $Ax \le b$, $a^T x \le \beta$ T.D.I and a integral $\Rightarrow Ax \le b$, $a^T x = \beta$ T.D.I.