Totally unimodular matrices

Def: A matrix A is *totally unimodular* if each sub-determinant of A is $0, +1, -1$ $\quad (\Rightarrow A \in \{0,1,-1\}^{m \times n})$

Theorem: $A \in \mathbb{Z}^{m \times n}$ unimodular $\Leftrightarrow$

$$P = \{x \mid Ax \le b,\ x \ge 0\} \text{ integral } \forall b \in \mathbb{Z}^m.$$

Proof:

"$\Rightarrow$" Let A totally unimodular, $b \in \mathbb{Z}^m$ and x a vertex of P. x is the unique solution of $A'x = b'$ for some subsystem $A'x \le b'$ of $\begin{pmatrix} A \\ -I \end{pmatrix} \cdot x \le \begin{pmatrix} b \\ 0 \end{pmatrix}$.

A totally unimodular $\Rightarrow \det A' \in \{1,-1\}$

$$\begin{pmatrix} A \\ -I \end{pmatrix} = \begin{pmatrix} & A & \\ -1 & & \\ & \ddots & \\ & & -1 \end{pmatrix}$$

$$\Rightarrow x = \underbrace{(A')^{-1}}_{\text{integral}} \cdot \underbrace{b'}_{\text{integral.}} \text{ integral}$$

"$\Leftarrow$" Suppose that all vertices of P are integral $\forall b \in \mathbb{Z}^m$. Let $A' \in \mathbb{Z}^{k \times k}$ nonsingular submatrix of A, w.l.o.g. $A = \left(\begin{array}{c|c} A' & * \\ \hline * & * \end{array}\right)$

$$(A \ \ I_m) = \begin{array}{cccc} * & \circ & \circ & * \\ \multicolumn{4}{c}{\left(\begin{array}{c|c|c|c} A' & * & I_k & 0 \\ \hline * & * & 0 & I_{m-k} \end{array}\right)^{z'}} \end{array} \qquad \det B = \det A'$$

$B \in \mathbb{Z}^{m \times m}$

Show that B^{-1} is integral $\Big(\underbrace{\det(B)}_{\text{integral}} \cdot \underbrace{\det(B^{-1})}_{\text{integral}} = 1 \Rightarrow |\det(B)| = 1 \Big)$

Let $i \in \{1, \dots, m\}$ and show that $B^{-1} e_i \in \mathbb{Z}^m$

Let $y \in \mathbb{Z}^m$ such that $z := y + B^{-1} e_i \geq 0$

Show that z is integral.

$$b := B \cdot z = B \cdot y + e_i \in \mathbb{Z}^m$$

Add zeros to z $\to$ z' with $(A, I_m) \cdot z' = B \cdot z = b$

$\to$ z'' consisting of the first n entries of z' belongs to P.

z'' satisfies $\begin{pmatrix} A \\ -I \end{pmatrix} \cdot z'' \leq \begin{pmatrix} b \\ 0 \end{pmatrix}$ with equality for the first k rows and the last $n-k$ rows.

$\to$ z'' is a vertex of P.

$\to$ $z'' \in \mathbb{Z}^n$ $\to$ $z' \in \mathbb{Z}^{n+m}$ $\to$ z integral. $\square$

<u>Theorem:</u> Let $A \in \mathbb{Z}^{m \times n}$. The following statements are equivalent:

(i) A is totally unimodular

(ii) $\forall b \in \mathbb{Z}^m, \forall c \in \mathbb{Z}^n$

$\max\{c^T x \mid Ax \leq b, x \geq 0\} = \min\{y^T b \mid y^T A \geq c^T, y \geq 0\}$

have integral optimal solutions x and y (if finite)

(iii) $Ax \leq b, x \geq 0$ is TDI for all $b \in \mathbb{R}^m$

(iv) $\forall R \subseteq \{1, \dots, m\} \; \exists$ partition $R = R_1 \dot\cup R_2$:

$$\sum_{i \in R_1} a_{ij} - \sum_{i \in R_2} a_{ij} \in \{-1, 0, 1\} \quad \forall j = 1, \dots, n$$

<u>Corollary:</u>

(i) The incidence matrix of an undirected graph is totally unimodular $\Leftrightarrow$ graph is bipartite.

(ii) The incidence matrix of any digraph is totally unimodular.