

# Totally unimodular matrices

Def: A matrix  $A$  is totally unimodular if each sub-determinant of  $A$  is  $0, +1, -1$  ( $\Rightarrow A \in \{0, 1, -1\}^{m \times n}$ )

Theorem:  $A \in \mathbb{Z}^{m \times n}$  unimodular  $\Leftrightarrow$

$$P = \{x \mid Ax \leq b, x \geq 0\} \text{ integral } \forall b \in \mathbb{Z}^m.$$

Proof:

$\Rightarrow$  Let  $A$  totally unimodular,  $b \in \mathbb{Z}^m$  and  $x$  a vertex of  $P$ .  $x$  is the unique solution of  $A'x = b'$  for some subsystem  $A'x \leq b'$  of  $\begin{pmatrix} A \\ -I \end{pmatrix} \cdot x \leq \begin{pmatrix} b \\ 0 \end{pmatrix}$ .

$A$  totally unimodular  $\Rightarrow \det A' \in \{1, -1\}$

$$\begin{pmatrix} A \\ -I \end{pmatrix} = \begin{pmatrix} A & & \\ & \dots & \\ & & -1 \end{pmatrix}$$

$$\Rightarrow x = \underbrace{(A')^{-1}}_{\text{integral}} \cdot \underbrace{b'}_{\text{integral}} \text{ integral}$$

$\Leftarrow$  Suppose that all vertices of  $P$  are integral  $\forall b \in \mathbb{Z}^m$ . Let  $A' \in \mathbb{Z}^{k \times k}$  nonsingular submatrix of  $A$ , w.l.o.g.

$$A = \left( \begin{array}{c|c} A' & * \\ \hline * & * \end{array} \right)$$

$$(A \ I_m) = \left( \begin{array}{c|c|c|c} A' & * & I_k & 0 \\ \hline * & * & 0 & I_{m-k} \end{array} \right) \begin{matrix} \\ \\ \\ \end{matrix}$$

$B \in \mathbb{Z}^{m \times m}$

$$\det B = \det A'$$

Show that  $B^{-1}$  is integral ( $\frac{\det(B) \cdot \det(B^{-1})}{\text{integral} \cdot \text{integral}} = 1$   
 $\rightarrow |\det(B)| = 1$ )

Let  $i \in \{1, \dots, m\}$  and show that  $B^{-1} \cdot e_i \in \mathbb{Z}^m$

Let  $y \in \mathbb{Z}^m$  such that  $z := y + B^{-1}e_i \geq 0$   
 Show that  $z$  is integral.

$$b := B \cdot z = B \cdot y + e_i \in \mathbb{Z}^m$$

Add zeros to  $z \rightarrow z'$  with  $(A, I_m) \cdot z' = B \cdot z = b$

$\rightarrow z''$  consisting of the first  $n$  entries of  $z'$  belongs to  $P$ .

$z''$  satisfies  $\begin{pmatrix} A \\ -I \end{pmatrix} \cdot z'' \leq \begin{pmatrix} b \\ 0 \end{pmatrix}$  with equality for the first  $k$  rows and the last  $n-k$  rows.

$\rightarrow z''$  is a vertex of  $P$ .

$\rightarrow z'' \in \mathbb{Z}^n \rightarrow z' \in \mathbb{Z}^{n+m} \rightarrow z$  integral.  $\square$

Theorem: Let  $A \in \mathbb{Z}^{m \times n}$ . The following statements are equivalent:

(i)  $A$  is totally unimodular

(ii)  $\forall b \in \mathbb{Z}^m, \forall c \in \mathbb{Z}^n$

$\max \{ c^T x \mid Ax \leq b, x \geq 0 \} = \min \{ y^T b \mid y^T A \geq c^T, y \geq 0 \}$   
 have integral optimal solutions  $x$  and  $y$  (if finite)

(iii)  $Ax \leq b, x \geq 0$  is TDI for all  $b \in \mathbb{R}^m$

(iv)  $\forall R \subseteq \{1, \dots, m\} \exists$  partition  $R = R_1 \cup R_2$  :

$$\sum_{i \in R_1} a_{ij} - \sum_{i \in R_2} a_{ij} \in \{-1, 0, 1\} \quad \forall j = 1, \dots, n$$

Corollary:

(i) The incidence matrix of an undirected graph is totally unimodular  $\Leftrightarrow$  graph is bipartite.

(ii) The incidence matrix of any digraph is totally unimodular.