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# Exercise V

**Problem 1** (again)
Let $\mathcal{T} \subseteq \{\text{true}, \text{false}\}^n$ be a set of truth assignments. Then there is a Boolean expression $\Phi$ such that $\mathcal{T}$ is exactly the set of truth assignments satisfying $\Phi$. We want to investigate under what conditions on $\mathcal{T}$ we can express $\Phi$ as a conjunction of Horn clauses.

Some notation: $T := T_1 \wedge T_2$ is the truth assignment $T(x) := T_1(x) \wedge T_2(x)$. And we define a partial order $\leq$ with $T_1 \leq T_2$ if $T_1(x) = \text{true}$ implies $T_2(x) = \text{true}$.

Now show the equivalence of the following three statements:

i) $\mathcal{T}$ is the set of truth assignments satisfying a conjunction of Horn clauses.

ii) If $T_1, T_2 \in \mathcal{T}$, then $T_1 \wedge T_2 \in \mathcal{T}$.

iii) If $T \notin \mathcal{T}$ (more precisely $T \in \{\text{true}, \text{false}\}^n \setminus \mathcal{T}$), then $S := \{T' \in \mathcal{T} : T \leq T'\}$ is either empty or there exists a unique $T^* \in S$ such that $T^* \leq T' \; \forall T' \in S$.

**Problem 2**
A Boolean function $f$ is called monotone if switching a variable from false to true cannot change the result of $f$ from true to false. Show that a Boolean function $f$ is monotone if and only if it can be expressed by a Boolean circuit without negations.

**Problem 3**
Consider a vocabulary with a constant $c$, variables $x, y, z_n$, $n$-ary functions $f_n$, and $n$-ary relations $R_n$. Formalize this as a vocabulary $\Sigma = (\Phi, \Pi, r)$ and a set of vocabulary $V$.
Which of the following strings are first-order logic terms or expressions for this vocabulary?

i) $\underline{f_3(f_0, c, x)}$

ii) $\underline{(\neg x \vee y)}$

iii) $\underline{\forall x \forall y R_2(x, y)}$

iv) $\underline{\forall x \forall x = (f_2(x, x), x)}$

v) $\underline{\phi = R_n(z_1, z_2, \ldots, z_n)}$

vi) $\underline{(R_1(f_1(x)) \wedge \neg R_1(f_1(x)))}$

**Problem 4**
Try to find first-order logic expressions for the following statements ...

a) ... from number theory:

i) Every $x$ has a successor $x + 1$.

ii) All numbers are equal.

iii) If there is a $x$ such that $x$ is the successor of every element, then all elements are equal.

b) ... from graph theory:

i) The graph is symmetric.

ii) There is a path from $x$ to $y$.