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Complexity theory and fixed parameter algorithms
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Exercise VI

Throughout this exercise, x and y are variables, and ϕ and ψ denote expressions.

Problem 1
Give an example for the following statements or show that they are impossible to satisfy:

i) x is free in expression ϕ, but has a bound occurrence in ϕ.

ii) x is free and bound in ϕ.

iii) x is not free and not bound in ϕ.

Problem 2
Prove the following statements about quantifiers.

i) If ϕ is valid, then $\forall x \phi$ is valid.

ii) If x is not free in ϕ, then $\phi \Rightarrow \forall x \phi$ is valid.

iii) For all ϕ and ψ, $(\forall x(\phi \Rightarrow \psi)) \Rightarrow ((\forall x \phi) \Rightarrow (\forall x \psi))$ is valid.

Problem 3
An expression is said to be in *prenex normal form* if it consists of a sequence of quantifiers ($\forall$ or $\exists$) followed by an expression without any quantifiers.
Prove the following statements, which help to establish that any expression can be equivalently written in prenex normal form. ($\phi \equiv \psi$ simply means $\phi \iff \psi$ but also suggests a proof.)

i) $\forall x(\phi \wedge \psi) \equiv (\forall x \phi \wedge \forall x \psi)$

ii) If x is not free in ψ, then $\forall x(\phi \wedge \psi) \equiv (\forall x \phi \wedge \psi)$.

iii) If x is not free in ψ, then $\forall x(\phi \vee \psi) \equiv (\forall x \phi \vee \psi)$.

iv) If y does not appear in ϕ, then $\forall x \phi \equiv \forall y \phi[x \leftarrow y]$.

Problem 4
We discussed some limits of the first-order logic vocabulary for graph theory last time: The inability to express "There is a path from x to y". We also talked about a way to circumvent this with a relation $P(x, y) = \{(x, y) :$ there is a path from x to $y\}$. Still we want to use this relation without relying too much on the model.
Suppose the universe not only includes nodes, but also all subsets of nodes of the graph. Suppose further, that the vocabulary contains a unary relation S which contains exactly the sets. Finally, there is a binary relation $\in$ which tests whether $x \in y$.

i) Can you derive P with this?

ii) How would you have to use P in a first-order logic expression?

iii) This moves the problem to S and $\in$. Have we gained anything by doing so? Can we push this further?