TU Berlin
Department of Mathematics
Prof. Dr. Martin Skutella
Daniel Dressler

Complexity theory and fixed parameter algorithms
Winter term 2008/9

Exercise IX

Problem 1

Show that the field of real numbers $\mathbb{R}$ cannot be described in first-order logic.
More precisely: Show that there is no *countable* set of first-order axioms such that every model satisfying these axioms is isomorphic to $(\mathbb{R}, 0, 1, +, \cdot, <)$.

Problem 2

It is known, that the set $\{\Phi : \mathbb{R} \models \Phi\}$ is decidable for the vocabulary $0, 1, +, \cdot, <$. For a proof, we will restrict us to the case where multiplication is restricted to produce *linear terms only.*

Let Φ be a sentence in prenex normal form such that $\mathbb{R} \models \Phi$. So Φ looks like

$$\underline{Q_1 x_1 Q_2 x_2 \dots Q_n x_n \psi(x_1, \dots, x_n)}$$

where $Q_i \in \{\exists, \forall\}$ and ψ contains no quantifiers.
We want to come up with a way to remove $Q_n x_n$ from Φ to obtain a sentence

$$\Phi' = \underline{Q_1 x_1 Q_2 x_2 \dots Q_{n-1} x_{n-1} \psi'(x_1, \dots, x_{n-1})}.$$

For this:

a) Show that $\psi(x_1, \dots, x_n)$ is a Boolean combination of k atomic expressions that can be written as

$$\underline{x_n \gtreqless l_i(x_1, \dots, x_{n-1})},$$

where the $l_i, 1 \le i \le k$, are *affine functions with rational coefficients*, and some more atomic expressions that do not contain x_n.

b) Assume $Q_n = \forall$. Show that ψ' can be taken to be $\underline{\bigwedge_{t \in T} \psi[x_n \leftarrow t]}$ where T is the set containing the following terms:

   i) $\underline{l_i(x_1, \dots, x_{n-1})}$, $1 \le i \le k$,
   ii) $\underline{\frac{1}{2}(l_i(x_1, \dots, x_{n-1}) + l_j(x_1, \dots, x_{n-1}))}$, $1 \le i < j \le k$,
   iii) $\underline{l_i(x_1, \dots, x_{n-1}) - 1}$, $1 \le i \le k$,
   iv) $\underline{l_i(x_1, \dots, x_{n-1}) + 1}$, $1 \le i \le k$.

c) Now consider $Q_n = \exists$. Reduce this to the case above.

d) Finish the induction.

e) Conclude that $\{\Phi : \mathbb{R} \models \Phi\}$ with this restricted multiplication can be decided (in double-exponential time).