ADM III: Linear and convex optimization in game theory,
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Assignment 1

— Solutions —

Exercise 1. Suppose two players argue over an object whose value to player P_i is $v_i > 0$ (for $i = 1,2$). Time is modeled as a continous variable that starts at 0 and runs indefinetly. Each player chooses when to concede the object to the other player; if the first player to concede does so at time t, the other player obtains the object at that time. In case of ties, the object is split evenly among the players. Each half of the object is worth $\frac{v_i}{2}$ to player P_i. Since time is valuable, it makes no sense to wait arbitrary long: until the first concession, each player looses one unit of payoff per unit of time.

Formulate the situation as a strategic game and show that in all Nash equilibria one player concedes immediatly.

Solution: *The set of strategies of player P_i is $X_i = [0,\infty)$ and his payoff function is*

$$u_i(t_1,t_2) = \begin{cases} -t_i & \text{if } t_i < t_j \\ \frac{v_i}{2} - t_i & \text{if } t_i = t_j \\ v_i - t_j & \text{if } t_j < t_i \end{cases}$$

where $j \in \{1,2\} \setminus \{i\}$. Let (t_1,t_2) be a pair of strategies chosen by P_1 and P_2.

- *If $t_1 = t_2$, then each player can increase his payoff by waiting a little longer. Thus, this is not a Nash equilibrium.*
- *If $0 < t_1 < t_2$, then P_1 can increase his payoff to zero by deviating to $t_1 = 0$.*
- *If $0 = t_1 < t_2$, then P_1 can increase his payoff by deviating to a time slightly later than t_2, unless $v_1 - t_2 \leq 0$.*
- *Similiar for player P_2.*

Therefore, (t_1,t_2) is a Nash equilibrium iff

$$\text{either } 0 = t_1 < t_2 \text{ and } t_2 \geq v_1, \text{ or } 0 = t_2 < t_1 \text{ and } t_1 \geq v_2.$$

Note that this is independent of the players' valuation!

$\Diamond$

Exercise 2. Consider the following payoff-matrices of two-player games:

a)

$$\begin{bmatrix} (3,1) & (4,1) & (5,9) \\ (2,6) & (5,3) & (6,3) \\ (1,4) & (1,5) & (9,2) \end{bmatrix}$$

b)

$$\begin{bmatrix} (3,1) & (4,0) & (4,1) \\ (4,0) & (3,0) & (3,1) \\ (4,1) & (3,0) & (3,1) \end{bmatrix}$$

Find the Nash equilibria of the games (if exist).

Solution: *Let $X = \{a,b,c\}$ and $Y = \{a',b',c'\}$. Recall that a profile $(x^*,y^*) \in X \times Y$ is a NE iff*

$$(x^*,y^*) \in B(x^*,y^*) := B_1(y^*) \times B_2(x^*).$$

Let us construct the (set-valued) matrices with entries $\{B(x,y)\}_{x\in X, y\in Y}$:

a)

$$\begin{bmatrix} (a,c') & (b,c') & (c,c') \\ (a,a') & (b,a') & (c,a') \\ (a,b') & (b,b') & (c,b') \end{bmatrix}$$

Thus, the game has no NE.

b)

$$\begin{bmatrix} \{b,c\}\times\{a',c'\} & \{a\}\times\{a',c'\} & \{a\}\times\{a',c'\} \\ \{b,c\}\times\{c'\} & \{a\}\times\{c'\} & \{a\}\times\{c'\} \\ \{b,c\}\times\{a',c'\} & \{a\}\times\{a',c'\} & \{a\}\times\{a',c'\} \end{bmatrix}$$

Thus, the only NE are (a,c') and (c,a').

$\Diamond$

Exercise 3. Give an example of a preference relation $\preceq$ on a set of strategies X for which it is not possible to determine a payoff function, i.e., a function $u : X \to \mathbb{R}$ satisfying

$$x \prec x' \quad \Longleftrightarrow \quad u(x) < u(x') \quad \forall x,x' \in X.$$

Solution: *Let $X = \{a,b,c\}$ where $b \prec c$ is the only preference relation. Then a and b are incomparable which implies $u(a) = u(b)$. Moreover, since a and c are incomparable as well, $u(a) = u(c)$ must also be true. Thus, $u(b) = u(c)$ follows in contradiction to $b \prec c$.*

$\Diamond$

Exercise 4. Consider $X = \mathbb{R}^2$ and the optimization problem

$$\begin{array}{lll} \max & x_1 + x_2 & \\ \text{s.t.} & x_1 + \frac{1}{2}x_2 & \leq 1 \\ & x_1 + 2x_2 & \leq 2. \end{array}$$

a) Let $M = \{(0,0),(1,0),(2,0),(\frac{1}{2},\frac{1}{2}),(\frac{2}{3},\frac{2}{3})\}$. Which elements in M are feasible for the problem?

b) Write down the dual linear program.

c) Determine the optimal solutions of the dual linear program.

d) Which elements in M are optimal solutions of the primal problem? (Proof!)

Hint to solve d): Use the result of c) to prove strong duality.

Solution:

a) $(2,0)$ *is the only element in M which is not feasible for the problem: the constraint* $x_1 + \frac{1}{2}x_2 \leq 1$ *is violated by* $(2,0)$.

b) The dual linear program is

$$\begin{array}{lll} \min & y_1 + 2y_2 & \\ s.t. & y_1 + y_2 & = 1 \\ & \frac{1}{2}y_1 + 2y_2 & = 1 \\ & y_1, y_2 & \geq 0. \end{array}$$

c) Solving the linear equality system shows that $(\frac{2}{3},\frac{1}{3})$ *is the unique solution of the problem with objective value* $\frac{4}{3}$.

d) Note that a feasible solution (x_1,x_2) *is optimal for the primal problem if its objective value* $x_1 + x_2$ *is exactly the objective value* $\frac{4}{3}$ *of the dual. Thus, the only optimal solution to the primal problem in M is* $(\frac{2}{3},\frac{2}{3})$.

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Exercise 5. Consider $X = \mathbb{R}^2$ and the optimization problem

$$\begin{array}{lll} \max & x_1 + x_2 & \\ \text{s.t.} & x_1 + \frac{1}{2}x_2 & \leq 1 \\ & x_1 + 2x_2 & \leq 2 \\ & x_1 + x_2 & \leq 4. \end{array}$$

a) Write down and solve the dual linear program.

b) Determine the Lagrange function.

c) Guess a feasible solution of the primal problem and check with the help of the complementary slackness conditions whether your guess is an optimal solution.

Solution:

a) The dual linear program is

$$\begin{array}{lll} \min & y_1+2y_2+4y_3 & \\ s.t. & y_1+y_2+y_3 & =1 \\ & \frac{1}{2}y_1+2y_2+y_3 & =1 \\ & y_1,y_2,y_3 & \geq 0. \end{array}$$

Note that

$$\left\{ \begin{pmatrix} \frac{2}{3} \\ \frac{1}{3} \\ 0 \end{pmatrix} + t \begin{pmatrix} -\frac{2}{3} \\ -\frac{1}{3} \\ 1 \end{pmatrix} \mid t \in \mathbb{R} \right\}$$

is the set of solutions of the equality system

$$\begin{array}{ll} y_1+y_2+y_3 & =1 \\ \frac{1}{2}y_1+2y_2+y_3 & =1. \end{array}$$

Since $y_1,y_2,y_3 \geq 0$ iff $0 \leq t \leq 1$, the set of feasible solutions of the dual problem is therefore

$$\mathscr{L} := \left\{ \begin{pmatrix} \frac{2}{3} \\ \frac{1}{3} \\ 0 \end{pmatrix} + t \begin{pmatrix} -\frac{2}{3} \\ -\frac{1}{3} \\ 1 \end{pmatrix} \mid 0 \leq t \leq 1 \right\}.$$

Since

$$\min_{y \in \mathscr{L}} y_1+2y_2+4y_3 = \min_{0 \leq t \leq 1} \frac{4}{3} - \frac{8}{3}t$$

it follows that the optimal value is obtained for $t=1$, i.e., $y=(0,0,1)^T$ is the unique optimal solution of the dual problem with objective value 4*.*

b) Define the functions

$$\begin{array}{lll} g_1(x) & := & x_1+\frac{1}{2}x_2-1 \\ g_2(x) & := & x_1+2x_2-2 \\ g_3(x) & := & x_1+x_2-4. \end{array}$$

Then $x \in \mathbb{R}^2$ is feasible iff $g_i(x) \geq 0$ for $i \in \{1,2,3\}$. The corresponding Lagrange function is

$$L(x,y) = x_1+x_2-y_1(x_1-\frac{1}{2}x_2+1)-y_2(-x_1-2x_2+2)-y_3(-x_1-x_2+4).$$

c) *Recall the complemenetary slackness conditions*

$$\begin{aligned} y_i > 0 &\implies g_i(x) = 0 \\ g_i(x) < 0 &\implies y_i = 0. \end{aligned}$$

We know from $a)$ *that* $y = (\frac{2}{3}, \frac{1}{3}, 0)^T$ *is the unique optimal solution of the dual problem. Thus, a feasible solution* (x_1, x_2) *of the primal problem is optimal if the first and second constraint are satisfied with equality. This is true for* $(x_1, x_2) = (\frac{2}{3}, \frac{2}{3})$ *which is therefore optimal.*

$\diamondsuit$