ADM III: Linear and convex optimization in game theory,
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Assignment 2

— Solutions —

Exercise 1. Let Γ_a and Γ_b be two zero-sum matrix games which are defined by the following two payoff matrices of the row player (i.e., the first player).

a)

$$\begin{bmatrix} 1 & 0 \\ -2 & 4 \end{bmatrix}$$

b)

$$\begin{bmatrix} 1 & 3 \\ 2 & 0 \end{bmatrix}$$

Determine the optimal maxmin-strategy for the first player in the corresponding randomized matrix games.

Solution: *In general, if $A = (a_{ij}) \in \mathbb{R}^{m \times n}$ is the payoff matrix of the first player in a zero-sum game, the optimal maxmin-strategy (and therefore, P_1's strategy in a NE) is an optimal solution of*

$$\begin{aligned} &\max z \\ &\quad \sum_{i=1}^{m} a_{ij} x_i \geq z \quad \forall j = 1, \ldots, m \\ &\quad x \in \Delta_m. \end{aligned}$$

In case of (2×2)-matrices, we have $(x_1, x_2) \in \Delta_2$ iff $x_2 = 1 - x_1$.

a) In this special case, we obtain an optimal strategy for P_1 by solving

$$\begin{aligned} &\max_{x \geq 0} z \\ &\quad x - 2(1-x) \geq z \\ &\quad 4(1-x) \geq z. \end{aligned}$$

with unique optimal solution $x = \frac{6}{7}$. Thus, $(\frac{6}{7}, \frac{1}{7})$ is an optimal strategy for player 1.

b) Here, we need to solve

$$\max_{x\geq 0} z$$
$$x+2(1-x)\geq z$$
$$3x \geq z.$$

with unique optimal solution $\frac{1}{2}$. *Thus,* $(\frac{1}{2},\frac{1}{2})$. *is an optimal strategy for* P_1.

$\Diamond$

Exercise 2. Let $A=(a_{ij})\in\mathbb{R}^{m\times n}$ be the payoff matrix of the row player in a zero-sum matrix game. We say that

- row i_1 *dominates* row i_2 if $a_{i_1 j}\geq a_{i_2 j}$ holds for all columns j, and
- column j_1 *dominates* column j_2 if $a_{ij_1}\leq a_{ij_2}$ holds for all rows i.

a) Show that in a randomized matrix game, dominated rows and columns can be ignored when optimal maxmin-strategies are to be calculated.

b) Determine an optimal maxmin-strategy for the row player in the randomized zero-sum matrix game with payoff matrix

$$\begin{bmatrix} 2 & 1 & 1 & 2 \\ 1 & 2 & 0 & 2 \\ 0 & 3 & 4 & 4 \\ 1 & 3 & 5 & 4 \end{bmatrix}.$$

Solution:

a) Suppose row i_1 *dominates row* i_2 *and consider an optimal solution* x^* *of*

$$\max z$$
$$\sum_{i=1}^{m} a_{ij}x_i \geq z \quad \forall j=1,\ldots,m$$
$$x\in\Delta_m.$$

with objective value z^*. *Construct the vector* $\hat{x}$ *with*

$$\hat{x}_{i_2}=0, \quad \hat{x}_{i_1}=x^*_{i_1}+x^*_{i_2} \quad \textit{and} \quad \hat{x}_i=x^*_i \quad \forall i\neq i_1,i_2.$$

Since $a_{i_1 j}\geq a_{i_2 j}$ *holds for all columns* j, *, we have that*

$$\sum_{j=1}^{n} a_{ij}\hat{x}_i \geq \sum_{j=1}^{n} a_{ij}x^*_i \quad \forall i=1,\ldots,m.$$

Thus, $\hat{x}$ is a feasible solution and even optimal since the objecitve values are identical.

Now suppose that column j_1 dominates column j_2. Note that the linear program above is equivalent to the problem

$$\max_{x\in\Delta_m} \min_{j=1,\dots,n} \underbrace{\sum_{i=1}^{m} a_{ij}x_i}_{=:z_j}.$$

Since $a_{ij_1} \le a_{ij_2}$ holds for all rows i, we have for all $x \in \Delta_m$ that

$$z_{j_1} = \sum_{i=1}^{m} a_{ij_1}x_i \le z_{j_2} = \sum_{i=1}^{m} a_{ij_2}x_i.$$

It follows that

$$\min_j z_j = \min_{j\neq j_2} z_j,$$

i.e., we get the same solution if we solve the problem without column j_2.

b) *We can reduce the matrix, since*

- ○ *row 4 dominates row 2 and row 3,*
- ○ *column 3 dominates column 3 and column 4.*

Removing the dominated rows and columns in that order yields the matrix

$$\begin{bmatrix} 2 & 1 \\ 1 & 3. \end{bmatrix}.$$

The corresponding optimization problem

$$\begin{aligned} \max_{x\geq 0}\ & z \\ & 2x + (1-x) \geq z \\ & x + 3(1-x) \geq z \end{aligned}$$

has optimal solution $x = \frac{2}{3}$. Thus, $(\frac{2}{3}, \frac{1}{3})$ is an optimal strategy for P_1.

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Exercise 3. Consider the facility location game illustrated in Figure 1.

a) Calculate an allocation vector $x \in \mathbb{R}^3$ in the core.

b) Extend the graph by adding a new facility such that the core of the modified game has an empty core.

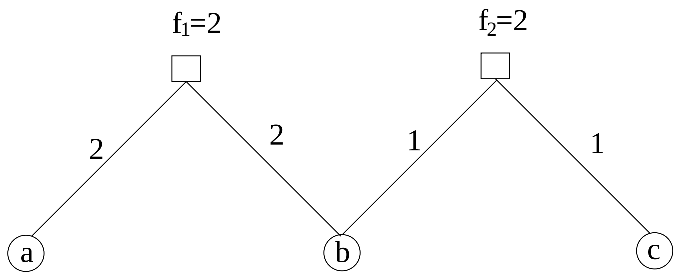

Figure 1: Facility location game with two facilities and three players $N = \{a, b, c\}$.

Solution:

a) *Any vector* $x \in \mathbb{R}^3$ *satisfying*

$$\begin{aligned} x_a + x_b + x_c &= 8 \\ x_a + x_b &\leq 6 \\ x_b + x_c &\leq 4 \\ x_a + x_c &\leq 7 \\ x_a &\leq 4 \\ x_b &\leq 4 \\ x_c &\leq 3 \end{aligned}$$

lies in the core of the game. For example $x = (4, 2, 2)^T$.

b) *Modify the game by adding a third facility as shown in Figure 2 Then the core is empty*

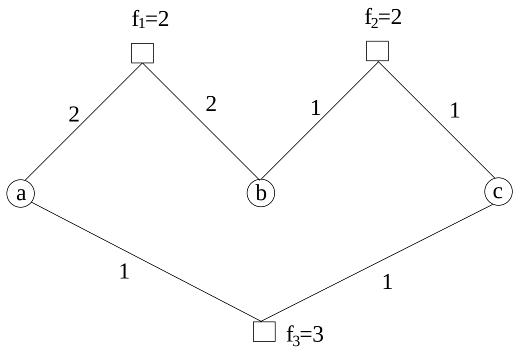

Figure 2: Facility location game with three facilities and three players $N = \{a, b, c\}$.

since any vector would need to satisfy the inequalities

$$\begin{aligned} x_a + x_b + x_c &= 8 \\ x_a + x_b &\leq 6 \\ x_b + x_c &\leq 4 \\ x_a + x_c &\leq 5. \end{aligned}$$

However, adding up the last three inequalities yields

$$x_a + x_b + x_c \leq 7.5 < 8$$

implying that there exists no solution of the inequality system.

$\Diamond$

Exercise 4. Let (N,c) be a cooperative cost game with $c(\emptyset) = 0$. We define the corresponding *dual payoff game* by

$$v(S) := c(N) - c(N \setminus S) \quad \forall S \subseteq N.$$

Show that both games have the same core.

Solution: *Suppose $x \in core(c)$. Then obviously,*

$$x(N) = c(N) = c(N) - c(N \setminus N) =: v(N).$$

Moreover, for each $S \subseteq N$ it follows that

$$v(S) = c(N) - c(N \setminus S) \geq x(N) - x(N \setminus S) = x(S),$$

since $x(N \setminus S) \leq c(N \setminus S)$ must be true. Thus, $x \in core(v)$ as well. Similiarly, if $x \in core(v)$, then

$$x(N) = v(N) = c(N) - c(N \setminus N) = c(N),$$

and for each $S \subseteq N$, we have

$$c(S) = c(N) - v(N \setminus S) \leq x(N) - x(N \setminus S) = x(S),$$

since $x(N \setminus S) \geq v(N \setminus S)$ holds. Thus, $x \in core(c)$. $\Diamond$