ADM III: Linear and convex optimization in game theory,
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Technische Universität Berlin
Institut für Mathematik
Dr. Britta Peis, Dr. Tobias Harks

Assignment 3

— Solutions —

Exercise 1. Let $\Gamma = (N, v)$ be a cooperative payoff game with $v : 2^N \to \mathbb{R}_+$. If v is not superadditive, a coalition might achieve a better payoff if it splits into disjoint subsets. Thus, it makes sense to consider the extended game $\tilde{\Gamma} = (N, \tilde{v})$ with

$$\tilde{v}(S) = \max\{\sum_i v(S_i) \mid S_i \subseteq S \text{ and } S_i \cap S_j = \emptyset \text{ if } i \neq j\} \quad \forall S \subseteq N.$$

a) Show that $\tilde{v}$ is monotone increasing and superadditive.

b) Show that if $v(N) = \tilde{v}(N)$, then core(v) = core$(\tilde{v})$.

Solution:

a) It follows by the definition that

$$S \subseteq T \quad \Longrightarrow \quad \tilde{v}(S) \leq \tilde{v}(T) \quad \textit{and}$$

$$\tilde{v}(S) + \tilde{v}(T) \leq \tilde{v}(S \cup T) \quad \forall S, T \subseteq N \textit{ with } S \cap T = \emptyset.$$

b) core$(\tilde{v}) \subseteq$ core(v) follows since $v \leq \tilde{v}$. To show the other direction, suppose that $x \in$ core(v). Since $v \geq 0$ it follows that $x \geq 0$. Consider an arbitrary coalition $S \subseteq N$ and let $\tilde{v}(S) = v(S_1) + \ldots + v(S_k)$ for pairwise disjoint subsets $S_1, \ldots, S_k$ of S. Then,

$$x(S_1 \cup \ldots \cup S_k) = \sum_{i=1}^{k} x(S_i) \geq \sum_{i=1}^{k} v(S_i) = \tilde{v}(S).$$

Thus, $x \in$ core$(\tilde{v})$.

◊

Exercise 2. Consider the game $\Gamma = (N, v)$ on three players $N = \{1, 2, 3\}$ whose payoff function is defined by $v(\{\}) = v(\{1\}) = v(\{2\}) = v(\{3\}) = 0, v(\{1, 2\}) = v(\{2, 3\}) = 1, v(\{1, 3\}) = 2$ and $v(\{1, 2, 3\}) = 4$. Determine all marginal vectors and the core of Γ.

Solution: *We can calculate the marginal vectors greedily for each permution π via*

$$x^{\pi}_{\pi_i} = v(\{\pi_1, \ldots, \pi_i\}) - v(\{\pi_1, \ldots, \pi_{i-1}\}) \quad \forall i = 1,2,3.$$

This way, we achieve the following marginal vectors:

π	x^{π}_1	x^{π}_2	x^{π}_3
123	0	1	4-1=3
132	0	2	4-2=2
213	1	0	3
231	3	0	1
312	2	2	0
321	3	1	0.

Since v is supermodular, we have

$$core(v) = conv\{x^{\pi} \mid \pi \text{ permuation of } N\}.$$

$\Diamond$

Exercise 3. Let $N = \{1,2,3\}$ and $v : 2^N \to \mathbb{R}_+$ be a payoff function with values

$$v(S) = \left\{ \begin{array}{ll} 0 & \text{if } |S| \leq 1 \\ 60 & \text{if } |S| = 2 \\ 72 & \text{if } |S| = 3. \end{array} \right\}$$

Determine the excesses with respect to the full allocations $x = (30,30,12)^T$ and $x' = (24,24,24)^T$. Is $l(x)$ lexicographically smaller than $l(x')$? Determine the nucleolus of $\Gamma(N,v)$.

Solution: *Note that $e(N,x) = e(N,x') = 0$ since x and x' are full allocations. For all proper coalitions $\emptyset \subset S \subset N$ we have*

$$e(S,x) = v(S) - x(S) = \left\{ \begin{array}{ll} 0 - 30 & \text{if } S \in \{\{1\},\{2\}\} \\ 0 - 12 & \text{if } S = \{3\} \\ 60 - 60 & \text{if } S = \{1,2\} \\ 60 - 42 & \text{if } S \in \{\{1,3\},\{2,3\}\} \end{array} \right\}$$

$$e(S,x') = v(S) - x'(S) = \left\{ \begin{array}{ll} 0 - 24 & \text{if } |S| = 1 \\ 60 - 48 & \text{if } |S| = 2 \end{array} \right\}$$

Thus,

$$l(x') = \begin{pmatrix} 12 \\ 12 \\ 12 \\ -24 \\ -24 \\ -24 \end{pmatrix} \prec l(x) = \begin{pmatrix} 18 \\ 18 \\ 0 \\ -12 \\ -30 \\ -30 \end{pmatrix}$$

It is not hard to see that x' is the nucleolus: Note the the linear program (LP.0)

$$
\begin{array}{lll}
\min_{x\geq 0} & \varepsilon & \\
s.t. & x_1+x_2+x_3 & =72 \\
 & \varepsilon+x_1+x_2 & \geq 60 \\
 & \varepsilon+x_2+x_3 & \geq 60 \\
 & \varepsilon+x_1+x_3 & \geq 60 \\
 & \varepsilon+x_i & \geq 0 \quad \forall i=1,2,3
\end{array}
$$

has the unique optimal solution $\varepsilon_0 = 12$, since summing up the three inequalities corresponding to two-element coalitions yields

$$2(x_1+x_2+x_3)+3\varepsilon \geq 180 \quad \textit{implies} \quad \varepsilon \geq \frac{1}{3}(180-2*72) = 12.$$

Thus, any optimal solution x of $(LP.0)$ must satisfy

$$x_1+x_2 = x_2+x_3 = x_1+x_3 = 60-\varepsilon_0 = 48,$$

and therefore $x_1 = x_2 = x_3 = 12$. Hence, x' is the nucleolus. ♢

Exercise 4. Consider a parliament with three parties $\{P_1, P_2, P_3\}$, where the parties P_1 and P_2 own 10 seats each, while party P_3 owns 19 seats. A majority of seats is necessary for a coalition to form the regime. How much power has each party? I.e., what is the Shapley value for the game in which each coalition has value 1 if it is able to form the regime, and zero otherwise?

Solution: *Since the Shapley value is the average over all marginal vectors, we need to determine the marginal values x_i^π for each player P_i and each permution π of $\{1,2,3\}$. The marginal values for P_1 are*

π	x_1^π
123	$v(\{1\})-v(\emptyset)=0$
132	$v(\{1\})-v(\emptyset)=0$
213	$v(\{1,2\})-v(2)=1$
231	$v(\{1,2,3\})-v(2,3)=0$
312	$v(\{1,3\})-v(3)=1$
321	$v(\{1,2,3\})-v(2,3)=0.$

Thus, $\Phi_1(v) = \frac{2}{6} = \frac{1}{3}$. By symmetrie, it follows that $\Phi_2(v) = \frac{1}{3}$. Since the Shapley value is a full allocation, we have that

$$\Phi_3(v) = v(N) - \Phi_1(v) - \Phi_2(v) = 1 - \frac{4}{6} = \frac{2}{6} = \frac{1}{3}.$$

♢