Technische Universität Berlin
Fakultät II – Institut für Mathematik

Berlin Mathematical School
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Topology

Dr. Carsten Schultz

Axel Werner

Problem Set 10

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Exercise 63. **4 points**

Let $f, g : S^1 \to S^1$ be continuous. Show:

(a) If $\deg f = \deg g$ then $f \simeq g$.

Hint: Remember Exercise 58, (b) $\Rightarrow$ (c).

(b) $\deg(g \circ f) = \deg g \cdot \deg f$.

Exercise 64. **4 points**

Provide an example of pointed spaces $(X, x_0), (Y, y_0)$, and maps $f, g : (X, x_0) \to (Y, y_0)$ such that $f \simeq g$, and yet $f_\# \neq g_\#$ as maps $\pi_1(X, x_0) \to \pi_1(Y, y_0)$. (Note that $f \simeq g$ means that f and g are *freely* homotopic, that is, not necessarily relative to the base point x_0.)
Hint: Choose a space Y such that $\pi_1(Y, y_0)$ is non-abelian.

Exercise 65. **4 points**

Recall (see Exercise 55) that a space X is *k-connected*, if every continuous map $S^\ell \to X$, with $-1 \leq \ell \leq k$, can be extended to a continuous map $D^{\ell+1} \to X$.
Prove: If X is a k-connected space and $X \simeq Y$ then Y is k-connected.

Exercise 66. **4 points**

Show that the following two subspaces of $\mathbb{R}^3$ are homotopy-equivalent:

$$\begin{aligned} X &:= \left\{x \in \mathbb{R}^3 \mid ||x|| = 1\right\} \cup \left\{(x_1, x_2, 0) \mid x \in \mathbb{R}^2, ||x|| \leq 1\right\} \\ Y &:= \left\{x \in \mathbb{R}^3 \mid ||x - (0,0,-1)|| = 1\right\} \cup \left\{x \in \mathbb{R}^3 \mid ||x - (0,0,1)|| = 1\right\} \end{aligned}$$

You don't have to give explicit formulas and check continuity of all involved maps in detail if you give reasonable descriptions of the homotopy equivalences and the involved homotopies in words and drawings.

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Exercise 67. **(Tutorial)**

Let $f : S^1 \to S^1$ continuous and $\deg f \neq 0$. Show that f is surjective.

Exercise 68. **(Tutorial)**

Compute $\pi_1(S^1 \times S^1)$ with the theorem of Seifert and van Kampen.

Exercise 69. **(Tutorial)**

Let $i_n : \mathbb{R}P^n \to \mathbb{R}P^{n+1}$ denote the map induced by the inclusion $S^n \to S^{n+1}$. Use the Seifert-van Kampen theorem to show that for $n > 1$ the homomorphism $(i_n)_\# : \pi_1(\mathbb{R}P^n) \to \pi_1(\mathbb{R}P^{n+1})$ is an isomorphism.