Technische Universität Berlin
Fakultät II – Institut für Mathematik

Berlin Mathematical School
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Topology

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Problem Set 5

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Exercise 30. **6 points**

(a) Let (X, d) be a metric space with the induced topology. Show that

$$d'(x, y) := \min \{d(x, y), 1\}$$

defines a metric on X that induces the same topology as d.

(b) For every $n \in \mathbb{N}$ let (X_n, d_n) be a metric space with d_n bounded by 1. Show that

$$d(x, y) := \sum_{n=0}^{\infty} 2^{-n} d(x_n, y_n)$$

defines a metric on the product $\prod_{n \in \mathbb{N}} X_n$.

(c) Show that countable products of metrisable spaces are metrisable.

Exercise 31. **4 points**

Let $X := (\mathbb{R}, \mathcal{T})$ with

$$\mathcal{T} := \{O \setminus A \mid O, A \subseteq \mathbb{R}, O \text{ open w.r.t. the Euclidean topology}, A \text{ countable}\}.$$

Show:

(a) X is a topological space.

(b) X is T_2.

(c) X is not regular.

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Exercise 32. **6 points**

Let X be a normal space and A a closed subset of X. Show:

(a) If $f : A \to \mathbb{R}^n$ is continuous and bounded, then there exists a continuous extension of f to X.

(b) If $f : A \to S^n$ is continuous, then there is a neighbourhood U of A such that there exists a continuous extension $F : U \to S^n$ of f.

Hint: A suitable continuous function $\mathbb{R}^{n+1} \setminus \{0\} \to S^n$ might help...

(c) Give an example of a space X, a closed subset $A \subset X$ and a continuous function $f : A \to S^0$ that does not have a continuous extension $F : X \to S^0$.

Exercise 33. **(Tutorial)**

Let X be a topological space and D a dense subset of X. Show that if there exists a closed discrete subspace S of X with $|S| \geq |\mathcal{P}(D)|$ then X is not normal.

Exercise 34. **(Tutorial)**

Define on $\mathbb{R}$ the topology $\mathcal{T}$ which has the basis $\{[a, b) \mid a, b \in \mathbb{R}\}$.

(a) Show that $(\mathbb{R}, \mathcal{T})$ is normal.

(b) Show that $X := (\mathbb{R}, \mathcal{T}) \times (\mathbb{R}, \mathcal{T})$ is not normal.

Hint: Use Exercise 33 with $D = \mathbb{Q} \times \mathbb{Q}$ and $S = \{(x, -x) \mid x \in \mathbb{R}\}$.

Recall: A space X is T_0, if for every two distinct points in X there exists an open set which contains one of the points, but not the other.

Exercise 35. **(Tutorial)**

Show:

(a) Every finite T_1 space has the discrete topology.

(b) There are finite T_0 spaces that are not discrete.

Exercise 36. **(Tutorial)**

Let X be a T_2 space and $x_1, \ldots, x_n \in X$ distinct points. Show that there are disjoint neighbourhoods $U_1, \ldots, U_n$ with $x_i \in U_i$ for $1 \leq i \leq n$.