Technische Universität Berlin
Fakultät II – Institut für Mathematik

Berlin Mathematical School
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Topology

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Problem Set 8

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Exercise 50. **4 points**

Let X be a contractible space and Y a *retract* of X, that is, a subspace of X such that there exists a map $r : X \to Y$ with $r|_Y = \mathrm{id}_Y$. Prove that Y is contractible.

Exercise 51. **4 points**

Let X be a space. Show that $\mathsf{C}X$ is contractible.

Exercise 52. **4 points**

Let $p_n : S^n \to \mathbb{R}P^n$ the quotient map from the definition of the projective space $\mathbb{R}P^n$. Show that $\mathbb{R}P^0$ is a space containing only one point, and $\mathbb{R}P^{n+1} \approx \mathbb{R}P^n \cup_{p_n} D^{n+1}$.

Exercise 53. **4 points**

Consider $S^1 \times S^1 \subset S^1 \times D^2$ and the mappings

$$i : S^1 \times S^1 \to S^1 \times D^2 \qquad t : S^1 \times S^1 \to S^1 \times D^2$$
$$(x, y) \mapsto (x, y), \qquad (x, y) \mapsto (y, x).$$

Show:

(a) $S^1 \times D^2 \cup_i S^1 \times D^2 \approx S^1 \times S^2$,

(b) $S^1 \times D^2 \cup_t S^1 \times D^2 \approx S^3$.

Hint: Exercise 3.

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Exercise 54. **(Tutorial)**

Let X be the subspace of $\mathbb{R}^2$ defined by

$$X := \left(\left(\left\{\frac{1}{n} \mid n \in \mathbb{N} \setminus \{0\}\right\} \cup \{0\}\right) \times I\right) \cup \left(I \times \{1\}\right)$$

Let $x_0 := (0,0) \in X$. Show that $\mathrm{id}_X \simeq c_{x_0}$ but $\mathrm{id}_X \not\simeq c_{x_0}$ rel $\{x_0\}$.

Exercise 55. **(Tutorial)**

For $k \geq -1$, a space X is called *k-connected*, if every continuous map $S^\ell \to X$, $-1 \leq \ell \leq k$, can be extended to a continuous map $D^{\ell+1} \to X$.

(a) Show that if $X \neq \emptyset$ is contractible then X is k-connected for all k.

(b) For which k do you think S^n is k-connected?

Exercise 56. **(Tutorial)**

Let X, Y be spaces, $A \subset X$, $A \neq \emptyset$, $p : X \to X/A$ the quotient map, $\{x_0\} = p[A]$ and $y_0 \in Y$. Show that there is a bijection between the sets of equivalence classes

$$\{f : X \to Y \mid f \text{ continuous}, f[A] = \{y_0\}\} /_{f \sim g :\Leftrightarrow f \simeq g \,\mathrm{rel}\, A}$$

and

$$\{f : X/A \to Y \mid f \text{ continuous}, f(x_0) = y_0\} /_{f \sim g :\Leftrightarrow f \simeq g \,\mathrm{rel} \{x_0\}}.$$

Remark: Important special cases are $X = D^n$, $A = S^{n-1}$ and $X = I$, $A = \{0, 1\}$.