Technische Universität Berlin
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Topology

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Problem Set 9

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Exercise 57. **6 points**

Let $n \in \mathbb{N}$ and $p_n : S^n \to \mathbb{R}P^n$ the quotient map from the definition of $\mathbb{R}P^n$.

(a) Show that p_n is a covering map for $n \geq 1$.

(b) Suppose $n \geq 2$ and $x_0 = p_n(1, 0, \ldots, 0) \in \mathbb{R}P^n$. Let $\alpha := [p_n \circ w]$ with

$$w : I \to S^n,\ t \mapsto (\cos \pi t, \sin \pi t, 0, \ldots, 0)$$

Show that $\alpha \in \pi_1(\mathbb{R}P^n, x_0)$ with $\alpha \neq \mathsf{e}$ and $\alpha^2 = \mathsf{e}$ (where e is the neutral element of $\pi_1(\mathbb{R}P^n, x_0)$).

Remark: In (b) you showed that the mapping $\Phi : \mathbb{Z}_2 \to \pi_1(\mathbb{R}P^n, x_0), 0 \mapsto \gamma, 1 \mapsto \alpha$ is an injective homomorphism. One can further show that Φ is also surjective, which implies that $\pi_1(\mathbb{R}P^n, x_0)$ is isomorphic to $\mathbb{Z}_2$ (if $n \geq 2$).

Exercise 58. **6 points**

Let (X, x_0) be a pointed space, $f : (S^1, (1, 0)) \to (X, x_0)$ a continuous function and $\gamma \in \pi_1(S^1, (1, 0))$ the element $\gamma = [g]$ with $g(t) = (\cos 2\pi t, \sin 2\pi t)$. Show that the following statements are equivalent (where e is the neutral element of $\pi_1(X, x_0)$):

(a) $f_\#(\alpha) = \mathsf{e}$ for all $\alpha \in \pi_1(S^1, (1, 0))$,

(b) $f_\#(\gamma) = \mathsf{e}$,

(c) $f \simeq c_{x_0} \operatorname{rel} \{(1, 0)\}$,

(d) $f \simeq c_{x'}$ for some $x' \in X$,

(e) f can be extended to a continuous function $\hat{f} : D^2 \to X$.

Hint: Show the equivalences cyclically in the given order. For (b) $\Rightarrow$ (c) recall that $I/\{0, 1\} \approx S^1$. For (d) $\Rightarrow$ (e) consider the cone and the cylinder over S^1. For (e) $\Rightarrow$ (a) compute $\pi_1(D^2, (1, 0))$ and consider $f_\#$ and $\hat{f}_\#$.

PLEASE TURN OVER

Exercise 59. **4 points**

Let X and Y be spaces, $x_0 \in X$, $y_0 \in Y$ and $p^X : X \times Y \to X$ and $p^Y : X \times Y \to Y$ the projections. Show that

$$\begin{array}{rcl} \left(p^X_\#, p^Y_\#\right) \; : \; \pi_1(X \times Y, (x_0, y_0)) & \to & \pi_1(X, x_0) \times \pi_1(Y, y_0) \\ \alpha & \mapsto & \left(p^X_\#(\alpha), p^Y_\#(\alpha)\right) \end{array}$$

is an isomorphism.

❄ **Exercise 60.**

For $k \in \mathbb{Z}$ we define

$$\begin{gathered} f_k : S^1 \to \mathbb{R}^2 \setminus \{0\} \\ (\cos\phi, \sin\phi) \mapsto (\cos k\phi, \sin k\phi). \end{gathered}$$

These are well-defined smooth maps.
Consider the differential form

$$\omega \;=\; \frac{-y\,dx + x\,dy}{x^2 + y^2}$$

on $\mathbb{R}^2 \setminus \{0\}$ (or a suitable similar 1-form). Use Stoke's Theorem to prove the following propositions.

(a) If there is a smooth map $g : D^2 \to \mathbb{R}^2 \setminus \{0\}$ with $g|_{S^1} = f_k$, then $k = 0$.

(b) If there is a smooth map $F : S^1 \times I \to \mathbb{R}^2 \setminus \{0\}$ with $F(x, 0) = f_k(x)$ and $F(x, 1) = f_l(x)$ for all $x \in S^1$, then $k = l$.

Remark: Except possibly for base point issues, you should also be able to prove these propositions using our results on the fundamental group of the circle, even for continuous instead of smooth maps. The point here is to see how completely(?) different methods yield the same or similar results.

Exercise 61. **(Tutorial)**

What is the fundamental group of a torus? What can you say about the one of the figure 8?

Exercise 62. **(Tutorial)**

Show that a pointed space (X, x_0) has trivial fundamental group $\pi_1(X, x_0)$ if and only if for all paths $p, q : I \to X$ with $p(0) = q(0) = x_0$ und $p(1) = q(1)$ we have $p \simeq q \;\mathsf{rel}\; \{0, 1\}$.

Cheerful christmas and a great new year everyone!