Aufgabe 1:

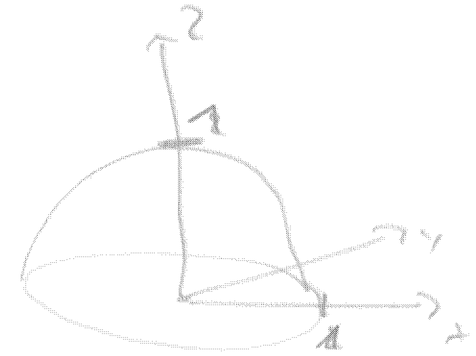
Es ist
$$\frac{\partial p}{\partial \phi}(\varphi,\vartheta) = \begin{pmatrix} -(R_2 - R_1\cos\vartheta)\sin\varphi \\ +(R_2 - R_1\cos\vartheta)\cos\varphi \\ 0 \end{pmatrix},$$
$$\frac{\partial p}{\partial \vartheta}(\varphi,\vartheta) = \begin{pmatrix} R_1\sin\vartheta\cos\varphi \\ R_1\sin\vartheta\sin\varphi \\ R_1\cos\vartheta \end{pmatrix},$$

also
$$N(p(\varphi,\vartheta)) = \frac{\partial p}{\partial \phi}(\varphi,\vartheta) \times \frac{\partial p}{\partial \vartheta}(\varphi,\vartheta)$$
$$= \begin{pmatrix} R_1\cos\vartheta\,(R_2 - R_1\cos\vartheta)\cos\varphi \\ R_1\cos\vartheta\,(R_2 - R_1\cos\vartheta)\sin\varphi \\ -R_1\sin\vartheta(R_2 - R_1\cos\vartheta)\sin^2\varphi - R_1\sin\vartheta(R_2 - R_1\cos\vartheta)\cos^2\varphi \end{pmatrix}$$
$$= \begin{pmatrix} R_1\cos\vartheta\,(R_2 - R_1\cos\vartheta)\cos\varphi \\ R_1\cos\vartheta\,(R_2 - R_1\cos\vartheta)\sin\varphi \\ -R_1\sin\vartheta(R_2 - R_1\cos\vartheta) \end{pmatrix},$$

also
$$|N(p(\varphi,\vartheta))| = \sqrt{R_1^2\cos^2\vartheta\,(R_2 - R_1\cos\vartheta)^2\cos^2\varphi + R_1^2\cos^2\vartheta\,(R_2 - R_1\cos\vartheta)^2\sin^2\varphi + R_1^2\sin^2\vartheta\,(R_2 - R_1\cos\vartheta)^2}$$
$$= R_1(R_2 - R_1\cos\vartheta)\underbrace{\sqrt{\cos^2\vartheta\cos^2\varphi + \cos^2\vartheta\sin^2\varphi + \sin^2\vartheta}}_{=1}$$
$$= R_1(R_2 - R_1\cos\vartheta),$$

Also: Fläche $A(\partial T)$:
$$A(\partial T) = \int_{\partial T} 1\,d\sigma = \int_{-\pi}^{\pi}\int_{-\pi}^{\pi} 1\cdot R_1(R_2 - R_1\cos\vartheta)\,d\varphi\,d\vartheta$$
$$= 2\pi R_1 \int_{-\pi}^{\pi}(R_2 - R_1\cos\vartheta)\,d\vartheta =$$
$$= 2\pi R_1\left(2\pi R_2 - R_1\sin\vartheta\Big|_{-\pi}^{\pi}\right) = \underline{\underline{4\pi^2 R_1 R_2}}$$

Aufgabe 2:



(i) Parametrisierung:

$$P : [0, 2\pi] \times [0, \tfrac{\pi}{2}] \to \mathbb{R}^3,$$

$$P(\varphi, \vartheta) = \begin{pmatrix} \cos\varphi \cos\vartheta \\ \sin\varphi \cos\vartheta \\ \sin\vartheta \end{pmatrix}.$$

Dann: $\dfrac{\partial P}{\partial \varphi}(\varphi, \vartheta) = \begin{pmatrix} -\sin\varphi \cos\vartheta \\ \cos\varphi \cos\vartheta \\ 0 \end{pmatrix}$, $\dfrac{\partial P}{\partial \vartheta}(\varphi, \vartheta) = \begin{pmatrix} -\cos\varphi \sin\vartheta \\ -\sin\varphi \sin\vartheta \\ \cos\vartheta \end{pmatrix}$

also $N(P(\varphi,\vartheta)) = \dfrac{\partial P}{\partial \varphi}(\varphi,\vartheta) \times \dfrac{\partial P}{\partial \vartheta}(\varphi,\vartheta) = \begin{pmatrix} \cos\varphi \cos^2\vartheta \\ \sin\varphi \cos^2\vartheta \\ \underbrace{\sin^2\varphi \cos\vartheta \sin\vartheta + \cos^2\varphi \sin\vartheta \cos\vartheta}_{= \cos\vartheta \sin\vartheta} \end{pmatrix}$

zeigt in der Tat „nach außen".

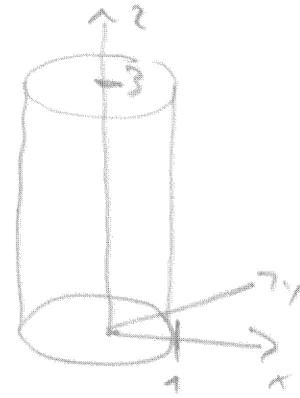
Also $\displaystyle\int_F v \, d\vec{\sigma} = \int_0^{2\pi} \int_0^{\pi/2} \langle v(P(\varphi,\vartheta)), N(P(\varphi,\vartheta)) \rangle \, d\vartheta \, d\varphi$

$$= \int_0^{2\pi} \int_0^{\pi/2} \begin{pmatrix} \cos\varphi \cos\vartheta \\ \sin\varphi \cos\vartheta \\ \sin\vartheta \end{pmatrix} \cdot \begin{pmatrix} \cos\varphi \cos^2\vartheta \\ \sin\varphi \cos^2\vartheta \\ \cos\vartheta \sin\vartheta \end{pmatrix} d\vartheta \, d\varphi$$

$$= \int_0^{2\pi} \int_0^{\pi/2} \cos^2\varphi \cos^3\vartheta + \sin^2\varphi \cos^3\vartheta + \sin^2\vartheta \cos\vartheta \, d\vartheta \, d\varphi$$

$$= \int_0^{2\pi} \int_0^{\pi/2} \cos\vartheta \, d\vartheta \, d\varphi = \int_0^{2\pi} \sin\vartheta \Big|_0^{\pi/2} d\varphi = \underline{\underline{2\pi}}$$

(ii) Parametrisierung:



$p: [0,2\pi] \times [0,3] \to \mathbb{R}^3$,

$p(\varphi,z) = \begin{pmatrix} \cos\varphi \\ \sin\varphi \\ z \end{pmatrix}$,

also $\frac{\partial p}{\partial \varphi}(\varphi,z) = \begin{pmatrix} -\sin\varphi \\ \cos\varphi \\ 0 \end{pmatrix}$, $\frac{\partial p}{\partial z}(\varphi,z) = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$, also

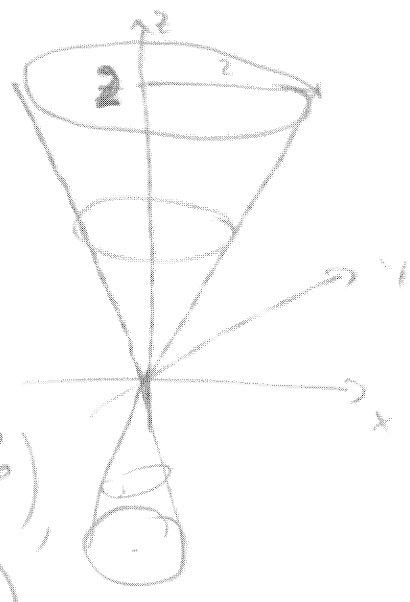
$N(p(\varphi,z)) = \begin{pmatrix} -\sin\varphi \\ \cos\varphi \\ 0 \end{pmatrix} \times \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} \cos\varphi \\ \sin\varphi \\ 0 \end{pmatrix}$.

zeigt „nach außen".

Also: $\int_F v \, d\vec{o} = \int_0^{2\pi} \int_0^3 \langle v(p(\varphi,z)), N(p(\varphi,z)) \rangle \, dz \, d\varphi$

$= \int_0^{2\pi} \int_0^3 \begin{pmatrix} \sin\varphi \\ -\cos\varphi \\ z^2 \end{pmatrix} \cdot \begin{pmatrix} \cos\varphi \\ \sin\varphi \\ 0 \end{pmatrix} dz \, d\varphi = \underline{\underline{0}}$

(iii) Parametrisierung:



$p: [0,2\pi] \times [-2,2] \to \mathbb{R}^3$

$p(\varphi,z) = \begin{pmatrix} z\cos\varphi \\ z\sin\varphi \\ z \end{pmatrix}$,

also $\frac{\partial p}{\partial \varphi}(\varphi,z) = \begin{pmatrix} -z\sin\varphi \\ z\cos\varphi \\ 0 \end{pmatrix}$, $\frac{\partial p}{\partial z}(\varphi,z) = \begin{pmatrix} \cos\varphi \\ \sin\varphi \\ 1 \end{pmatrix}$,

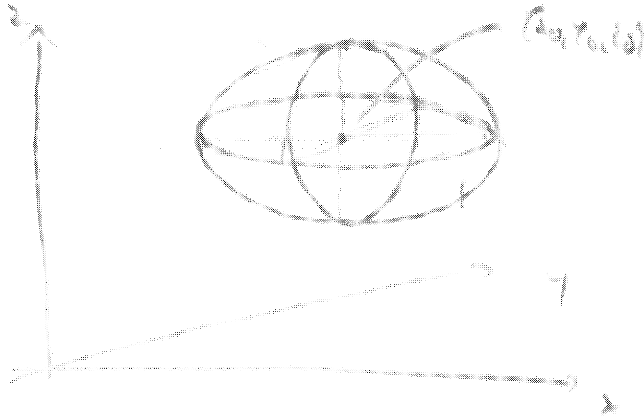
also $N(p(\varphi,z)) = \begin{pmatrix} z\cos\varphi \\ z\sin\varphi \\ -z\sin^2\varphi - z\cos^2\varphi \end{pmatrix} = \begin{pmatrix} z\cos\varphi \\ z\sin\varphi \\ -z \end{pmatrix}$

$\int_F v \, d\vec{o} = \int_0^{2\pi} \int_{-2}^{2} \begin{pmatrix} z\cos\varphi \\ z\sin\varphi \\ 3 \end{pmatrix} \cdot \begin{pmatrix} z\cos\varphi \\ z\sin\varphi \\ -z \end{pmatrix} dz \, d\varphi = \int_0^{2\pi} \int_{-2}^{2} z^2\cos^2\varphi + z^2\sin^2\varphi - 3z \, dz \, d\varphi$

$= \int_0^{2\pi} \int_{-2}^{2} z^2 - 3z \, dz \, d\varphi = 2\pi \int_{-2}^{2} z^2 - 3z \, dz = 2\pi \left(\frac{1}{3}z^3 - \frac{3}{2}z^2 \right)\Big|_{-2}^{2} = 2\pi\left(\frac{8}{3} + \frac{8}{3}\right) = \underline{\underline{\frac{32}{3}\pi}}$

Aufgabe 3:

(i) Ellipsoid um (x_0, y_0, z_0) mit Halbachsen a, b, c:

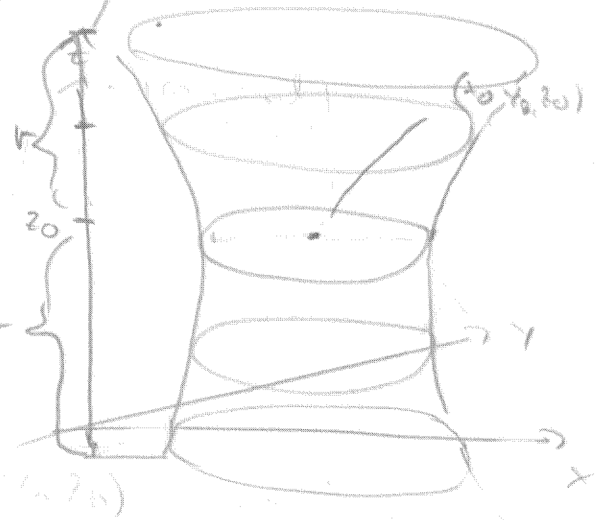


Parametrisierung:

$$P: [0, 2\pi] \times [-\pi/2, \pi/2] \to \mathbb{R}^3,$$

$$P(\varphi, \vartheta) = \begin{pmatrix} x_0 \\ y_0 \\ z_0 \end{pmatrix} + \begin{pmatrix} a \cos\varphi \cos\vartheta \\ b \sin\varphi \cos\vartheta \\ c \sin\vartheta \end{pmatrix}$$

(ii) elliptisches Hyperboloid:

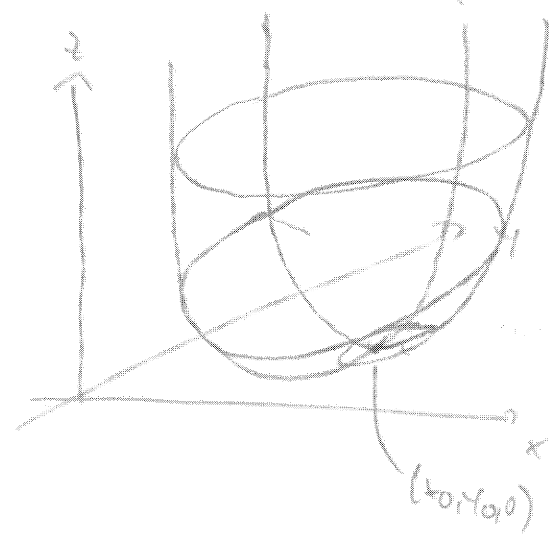


Parametrisierung:

$$P: [0, 2\pi] \times [-r, r] \to \mathbb{R}^3$$

$$P(\varphi, h) := \begin{pmatrix} x_0 \\ y_0 \\ z_0 \end{pmatrix} + \begin{pmatrix} a\sqrt{1 + h^2/c^2} \cdot \cos\varphi \\ b\sqrt{1 + h^2/c^2} \sin\varphi \\ h \end{pmatrix}.$$

(iii) elliptisches Paraboloid:



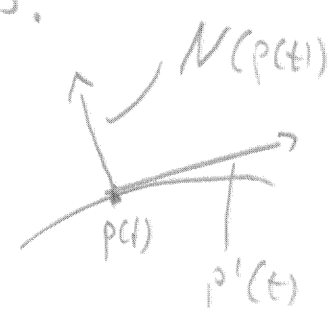
$$P: [0, 2\pi] \times [0, r] \to \mathbb{R}^3,$$

$$P(\varphi, z) = \begin{pmatrix} x_0 \\ y_0 \\ 0 \end{pmatrix} + \begin{pmatrix} a\sqrt{z} \cos\varphi \\ b\sqrt{z} \sin\varphi \\ z \end{pmatrix}.$$

Aufgabe 4:

Unter anderem: Ist $\mathcal{F} \subset \mathbb{R}^2$ quasireguläres Flächenstück und $M = [a,b] \subset \mathbb{R}$, $p: M \to \mathbb{R}^2$ eine quasireguläre Param., so ist $\mathcal{F}$ eine Kurve in unserem Sinn mit entspr. Param. p.

Ist $p'(t) = \begin{pmatrix} p_1'(t) \\ p_2'(t) \end{pmatrix}$,



so ist $N(p(t)) = \begin{pmatrix} -p_2'(t) \\ p_1'(t) \end{pmatrix}$,

also $|N(p(t))| = \sqrt{p_2'(t)^2 + p_1'(t)^2} = |p'(t)|$,

also für geeignetes $f: \mathbb{R}^2 \to \mathbb{R}$ z.B.

$$\int_{\mathcal{F}} f \, d\sigma = \int_a^b f(p(t)) \, |N(p(t))| \, dt = \int_a^b f(p(t)) \, |p'(t)| \, dt$$

$$= \int_{\mathcal{F}} f \, ds.$$

u.s.w.