Lösungsskizzen

Aufgabe 1:

(i) Ansatz: $\frac{1}{x(x+1)} = \frac{a}{x} + \frac{b}{x+1}$,

also $1 = a(x+1) + bx = x(a+b) + a$,

also $a = 1$ und $a+b = 0$,

also $\boxed{a = 1 \text{ und } b = -1}$

Somit ist $\int_1^2 \frac{1}{x(x+1)}\,dx = \int_1^2 \frac{1}{x}\,dx - \int_1^2 \frac{1}{x+1}\,dx$

$= \ln(x)\Big|_1^2 - \ln(x+1)\Big|_1^2$

$= \ln(2) - 0 - \ln(3) + \ln(2)$

$= \underline{2\ln(2) - \ln(3)}$

(ii) Polynomdivision:

$$
\begin{array}{l}
(2x^3 + 9x^2 + 8x + 5) : (x^2+4x+3) = 2x + 1 + \underbrace{\frac{2-2x}{x^2+4x+3}}_{\text{Rest}} \\
\underline{-(2x^3 + 8x^2 + 6x)} \\
\quad x^2 + 2x + 5 \\
\quad \underline{-(x^2 + 4x + 3)} \\
\qquad -2x + 2
\end{array}
$$

Nullstellen von x^2+4x+3 sind -1 und -3,

also $x^2+4x+3 = (x+1)(x+3)$

Ansatz: $\frac{2-2x}{x^2+4x+3} = \frac{a}{x+1} + \frac{b}{x+3}$, also

$2-2x = a(x+3) + b(x+1) = x(a+b) + (3a+b)$,

also $3a+b = 2$ und $a+b = -2$, also

$a = 2$ und $b = -4$.

Damit ist dann

$$\int \frac{2x^3+9x^2+8x+5}{x^2+4x+3}\,dx = \int (2x+1)\,dx + \int \frac{2}{x+1}\,dx - \int \frac{2}{x+3}\,dx$$

$$= x^2+x+2\ln(x+1) - 2\ln(x+3)$$

$$= x^2+x+2\ln\left(\frac{x+1}{x+3}\right).$$

(3) $\int \frac{x^2+x+1}{x(x^2+1)}\,dx$

Achtung: komplexe Nullstellen von $x(x^2+1)$,

nämlich: $a_1 = 0$
$a_2 = i$
$a_3 = -i$.

Trotzdem geht der Ansatz:

$$\frac{x^2+x+1}{x(x^2+1)} = \frac{x^2+x+1}{x(x-i)(x+i)} = \frac{a}{x} + \underbrace{\frac{b}{x-i} + \frac{c}{x+i}} = \frac{a}{x} + \frac{b(x+i)+c(x-i)}{(x-i)(x+i)}$$

diese können wir wieder zusammenfassen, um keine komplexen Zahlen im Nenner zu bekommen

$$= \frac{a}{x} + \frac{x(b+c)+i(b-c)}{x^2+1}$$

Also: $x^2+x+1 = a(x^2+1) + (x(b+c)+i(b-c))x$

$$= x^2(a+b+c) + xi(b-c) + a,$$

also
$$\left.\begin{aligned} a &= 1, \\ i(b-c) &= 1 \\ a+b+c &= 1 \end{aligned}\right\} \Rightarrow \begin{aligned} a &= 1, \\ b &= -i/2 \\ c &= i/2 \end{aligned},$$

also folgt

$$\int \frac{x^2+x+1}{x(x^2+1)}\,dx = \int \frac{1}{x}\,dx + \int \frac{0\cdot x+1}{x^2+1}\,dx = \int \frac{1}{x}\,dx + \int \frac{1}{1+x^2}\,dx$$

$$= \ln(x) + \arctan(x)\,dx.$$

<u>Aufgabe 2</u>: $f(x) = e^{x^2+y^2}$

$$\frac{\partial f}{\partial x}(x,y) = 2x\,e^{x^2+y^2}, \quad \frac{\partial f}{\partial y}(x,y) = 2y\,e^{x^2+y^2}.$$

$$g(x) = y\cos(xy) + 3x^2y$$

$$\frac{\partial g}{\partial x}(x,y) = -y\sin(xy)\cdot y + 6xy = 6xy - y^2\sin(xy)$$

$$\frac{\partial g}{\partial y}(x,y) = \cos(xy) - y\sin(xy)\cdot x + 3x^2 = $$
$$= 3x^2 + \cos(xy) - xy\sin(xy).$$

1. Weg: Ableitung aus Integral ziehen

$$\frac{d}{dy}\int_0^1 y\cos(xy) + 3x^2y\,dx = \int_0^1 \frac{\partial}{\partial y}\left(y\cos(xy) + 3x^2y\right)dx$$

$$= \int_0^1 3x^2 + \cos(xy) - xy\sin(xy)\,dx$$

$$= x^3\Big|_{x=0}^{x=1} + \frac{1}{y}\sin(xy)\Big|_{x=0}^{x=1} - y\underbrace{\int_0^1 x\sin(xy)\,dx}$$

$$= -\frac{x}{y}\cos(xy)\Big|_{x=0}^{x=1} + \frac{1}{y}\int_0^1\cos(xy)\,dx$$

$$= \frac{-\cos(y)}{y} + \frac{1}{y^2}\cdot\sin(xy)\Big|_{x=0}^{x=1}$$

$$= 1 + \frac{\sin(y)}{y} + \cos(y) - \frac{\sin(y)}{y} = \underline{\underline{1 + \cos(y)}}$$

2. Weg: direkt: $\displaystyle\int_0^1 y\cos(xy) + 3x^2y\,dx = y\int_0^1(\cos(xy) + 3x^2)\,dx$

$$= \sin(xy)\Big|_{x=0}^{x=1} + y\,x^3\Big|_{x=0}^{x=1} = \underline{\sin(y) + y}$$

Also $\displaystyle\frac{d}{dy}\int_0^1 y\cos(xy) + 3x^2y\,dx = \underline{\underline{\cos(y) + 1}}$

Aufgabe 3:

(i) Für $\varepsilon > 0$ ist $\int_\varepsilon^1 \ln(x)\,dx = \int_\varepsilon^1 1\cdot \ln(x) = x\cdot \ln x\Big|_\varepsilon^1 - \int_\varepsilon^1 x\cdot \frac{1}{x}\,dx$

$$= \varepsilon \ln(\varepsilon) - (1-\varepsilon).$$

und $\lim_{\varepsilon \downarrow 0} \varepsilon \cdot \ln(\varepsilon) = \lim_{\varepsilon\downarrow 0} \frac{\ln(\varepsilon)}{\frac{1}{\varepsilon}} \underset{\text{L'Hospital}}{=} \lim_{\varepsilon\downarrow 0} \frac{\frac{1}{\varepsilon}}{-\frac{1}{\varepsilon^2}}$

$$= \lim_{\varepsilon\downarrow 0}(-\varepsilon) = 0,$$

also existiert das uneigentliche Integral und es gilt:

$$\int_0^1 \ln(x)\,dx = \lim_{\varepsilon\downarrow 0}\int_\varepsilon^1 \ln(x) = \lim_{\varepsilon\downarrow 0}\left(\varepsilon\ln(\varepsilon) - (1-\varepsilon)\right)$$

$$= \underline{\underline{-1}}$$

(ii) $\int_{-\infty}^{\infty} \frac{x}{1+x^2}\,dx = \lim_{b\to\infty}\lim_{a\to-\infty}\int_a^b \frac{x}{1+x^2}\,dx.$

Es ist $\int_a^b \frac{x}{1+x^2}\,dx \underset{\substack{y=1+x^2\\ \frac{dy}{dx}=2x,\\ x\,dx=\frac{1}{2}dy}}{=\!=\!=} \frac{1}{2}\int_{1+a^2}^{1+b^2} \frac{1}{y}\,dy = \frac{1}{2}\ln(y)\Big|_{1+a^2}^{1+b^2}$

$$= \frac{1}{2}\left(\ln(1+b^2) - \ln(1+a^2)\right) = \frac{1}{2}\ln\left(\frac{1+b^2}{1+a^2}\right)$$

Aber schon $\lim_{a\to-\infty} \frac{1}{2}\ln\left(\frac{1+b^2}{1+a^2}\right)$ <u>existiert nicht</u>

(iii) $$\int_0^\infty \frac{e^{-2\sqrt{|x|}}}{\sqrt{|x|}}\,dx \overset{*}{=} \lim_{R\to\infty} \lim_{\varepsilon \downarrow 0} \int_\varepsilon^R \frac{e^{-2\sqrt{x}}}{\sqrt{x}}\,dx \qquad (x \geq 0)$$

$$\overset{t=\sqrt{x}}{=} \lim_{R\to\infty} \lim_{\varepsilon\downarrow 0} 2\int_\varepsilon^R e^{-2t}\,dt$$

$$\boxed{\begin{aligned} &\frac{dt}{dx} = \frac{1}{2\sqrt{x}} \\ &2dt = \frac{1}{\sqrt{x}}\,dx \end{aligned}}$$

$$= \lim_{R\to\infty} \lim_{\varepsilon\downarrow 0} \left(-e^{-2t}\right)\Big|_\varepsilon^R$$

$$= \lim_{R\to\infty} \lim_{\varepsilon\downarrow 0} \left(-e^{-2R} + e^{-2\varepsilon}\right)$$

$$= \lim_{R\to\infty} \left(-e^{-2R} + 1\right) = 0 + 1 = \underline{\underline{1}}$$

Aufgabe 4:

(f_n) konv. auf $[0,\infty)$ glm. gegen die Nullfunktion $f:[0,\infty)\to\mathbb{R}$, $f(x)=0$:

$$\|f_n - f\|_\infty = \sup_{x\in[0,\infty)} |f_n(x) - 0| = \sup_{x\in[0,\infty)} \underbrace{\left|\frac{n}{n^2+x^2}\right|}_{\leq \frac{n}{n^2} = \frac{1}{n}} \leq \frac{1}{n} \xrightarrow[n\to\infty]{} 0$$

Es ist aber

$$\int_0^\infty f_n(x)\,dx = \lim_{R\to\infty} \int_0^R \frac{n}{n^2+x^2}\,dx = \lim_{R\to\infty} \frac{1}{n}\int_0^R \frac{1}{1+(\frac{x}{n})^2}\,dx$$

$$= \lim_{R\to\infty} \int_0^{R/n} \frac{1}{1+t^2}\,dt = \lim_{R\to\infty} \arctan(t)\Big|_0^{R/n}$$

$\boxed{t = \frac{x}{n},\ \frac{dt}{dx} = \frac{1}{n},\ dt = \frac{1}{n}dx}$

$$= \lim_{R\to\infty} \arctan(R/n) = \underline{\underline{\pi/2}} \underset{n\to\infty}{\not\longrightarrow} 0 = \int_0^\infty f(x)\,dx$$