Technische Universität Berlin **WS 2010/2011**
Fakultät II – Institut f. Mathematik
Lecturer: Prof. Dr. J.-D. Deuschel hand out: Wed 05.01.2011
Assistant: Simon Wasserroth due: Wed 12.01.2011

Assignment 10
„Wahrscheinlichkeitstheorie 2 - Stochastic processes 1“

---

total points: 20 Points

Problem 1 **5 Points**

Let $\alpha > 0$, $X_0 \in \mathbb{R}$ be given and choose iid ϵ_i with zero mean and finite variance and continuous density. Define for $n \in \mathbb{N}_0$ the sequence $X_{n+1} = \alpha X_n + \epsilon_{n+1}$.

(i) Show that the (Markov-)process (X_n) is strong Feller.

(ii) Find a necessary condition for α for the process (X_n) to be stationary (if started in the stationary distribution).

Problem 2 **6 Points**

Let $(\Omega, \mathcal{F})$ be a measurable space and $T : \Omega \to \Omega$ a measurable map. Let $\mathcal{I}$ be the set of all invariant probability measures.

(i) Show that $\mathcal{I}$ is a convex set.

(ii) Show that a probability measure $\mu \in \mathcal{I}$ is extremal iff μ is ergodic. A measure is called extremal if for $\mu = \lambda\mu_1 + (1-\lambda)\mu_2$ with $\lambda \in [0, 1]$, $\mu_1, \mu_2 \in \mathcal{I}$ and $\mu_1 \neq \mu_2$ one gets $\lambda \in \{0, 1\}$.

Hint: For the first implication take some non ergodic measure μ such that there exists A with $\mu(A) = \lambda$ and construct an explicit convex combination of it.

For the converse implication assume that μ is ergodic and could be combinated as $\mu = \lambda\mu_1 + (1-\lambda)\mu_2$. Show that then also μ_1 and μ_2 must be ergodic and use the ergodic theorem.

Problem 3 **5 Points**

(i) Find a Markov process (in discrete time) that is not strong Feller.

(ii) Prove that every Markov chain on a finite state space has at least one ergodic stationary distribution.

(iii) Let $\mathbb{X}$ be an irreducible Markov chain on a finite state space E started in the invariant distribution p. Show that if $\mathbb{X}$ is periodic then p cannot be mixing. **Extra:** Show that if $\mathbb{X}$ is aperiodic then p is mixing.

Problem 4 **4 Points**

Let $\mathbb{P}$ be a measure on Ω, the path space of sequences $(x_0, x_1, x_2, \dots)$. Let F be a bounded real valued random variable on $(\Omega, \mathcal{F}, \mathbb{P})$. Define $f = \mathbb{E}\left[F \,|\sigma(x_0)\right]$. Let K be a Markov kernel (and the associated linear operator) with $K : \mathbb{R} \times \mathcal{F} \to [0, 1]$. T denotes the left shift on the path space. Show that then $Kf = \mathbb{E}\left[F \circ T \,|\sigma(x_0)\right]$.