Technische Universität Berlin **WS 2010/2011**
Fakultät II – Institut f. Mathematik
Lecturer: Prof. Dr. J.-D. Deuschel hand out: Wed 12.01.2011
Assistant: Simon Wasserroth due: Wed 19.01.2011

Assignment 11
„Wahrscheinlichkeitstheorie 2 - Stochastic processes 1“

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total points: 20 Points

Problem 1 **5 Points**
Let μ, ν be two probability measures on $(\Omega, \mathcal{F})$ with $\mu \ll \nu$ and density $\frac{\mathrm{d}\mu}{\mathrm{d}\nu} = f : \Omega \to \mathbb{R}$. Let $(\mathcal{F}_n)$ be a filtration and define μ_n, ν_n as the restriction of the measures to $\mathcal{F}_n$. Show that $\frac{\mathrm{d}\mu_n}{\mathrm{d}\nu_n} = \mathbb{E}_\nu[f|\mathcal{F}_n]$.

Problem 2 **5 Points**
Let (X_n) be a Markov chain on the state space $\{1, \ldots, N\}$ starting in $i \in \{1, \ldots, N\}$ with transition matrix $P \in \mathbb{R}^{N \times N}$. Assume that (X_n) is also a martingale (with respect to its own filtration). Show that then the states 1 and N must be absorbing, i.e. $P_{11} = P_{NN} = 1$.

Problem 3 **5 Points**
Let $(X_{n,i})_{n,i \in \mathbb{N}_0}$ be a family of independent $\mathbb{N}_0$ valued random variables with $\mathbb{P}(X_{n,i} = k) = p_k$ for all $k \in \mathbb{N}_0$ and $\mathbb{E}[X_{n,i}] = m$. Define the branching process (Y_n) by $Y_0 = 1$ and $Y_{n+1} = \sum_{i=1}^{Y_n} X_{n,i}$ for $n \in \mathbb{N}_0$ and $Z_n = \frac{1}{m^n} Y_n$.

(i) Show that (Z_n) is a martingale (with respect to its own filtration).

(ii) Assume $m = 1$ and $p_1 < 1$. Show that then Z_n converges to 0 almost surely but not in L^1.

Problem 4 **5 Points**
Let K be a Markov kernel on $\mathbb{R}$ and (X_n) the induced Markov process (in discrete time) starting in $x \in \mathbb{R}$. Assume that $\mathbb{E}[|X_n|] < \infty$. Assume that the measurable and bounded function $f : \mathbb{R} \to \mathbb{R}$ is harmonic, i.e. $Kf = f$. Show that then $(f(X_n))_n$ is a martingale.