Technische Universität Berlin **WS 2010/2011**
Fakultät II – Institut f. Mathematik
Lecturer: Prof. Dr. J.-D. Deuschel hand out: Wed 19.01.2011
Assistant: Simon Wasserroth due: Wed 26.01.2011

Assignment 12
„Wahrscheinlichkeitstheorie 2 - Stochastic processes 1“

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total points: 20 Points

Problem 1 — 5 Points

(i) Let σ, τ be two stopping times on the filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_n), \mathbb{P})$. Show that then:

$$\mathcal{F}_\tau \cap \mathcal{F}_\sigma = \mathcal{F}_{\sigma \wedge \tau}$$

(ii) Let (X_n) be an adapted process and τ an almost surely finite stopping time. Show that then X_τ is $\mathcal{F}_\tau$ measurable.

Problem 2 — 6 Points

Let (X_i) be iid random variables with $\mathbb{P}(X_1 = 1) = p$ and $\mathbb{P}(X_1 = -1) = 1 - p = q$ for some $p \in (0,1)$. Define the random walk (M_n) with $M_0 = 0$ and $M_n = \sum_{i=1}^n X_i$. Determine the probability for $\tau_a < \tau_b$ with $a, b \in \mathbb{Z}$, $a < 0 < b$, and $\tau_l = \inf\{k \in \mathbb{N} | M_k = l\}$.

Show first that $Z_n = \left(\frac{q}{p}\right)^{M_n}$ is a martingale. Then define $\tau_{a,b} = \inf\{n \in \mathbb{N}_0 | M_n = a \text{ or } M_n = b\}$ and apply then the optimal stopping theorem to the stopping time $\tau_{a,b} \wedge n$ and let $n \to \infty$. Observe that $\mathbb{P}(\tau_a < \tau_b) = \mathbb{P}(M_{\tau_{a,b}} = a)$.

Problem 3 — 5 Points

Let (M_n) denote the simple random walk. Define $\tau_{a,b} = \inf\{n \in \mathbb{N}_0 | M_n = a \text{ or } M_n = b\}$. Determine $\mathbb{E}[\tau_{a,b}]$ using the quadratic variation and optimal stopping.

Problem 4 — 4 Points

Define a Markov process $X = (R, B)$ on $\mathbb{N}^2$ starting in $X_0 = (1,1)$ with transition probabilities $\mathbb{P}((R_{n+1}, B_{n+1}) = (r+1, b) | (R_n, B_n) = (r, b)) = \frac{r}{r+b}$ and $\mathbb{P}((R_{n+1}, B_{n+1}) = (r, b+1) | (R_n, B_n) = (r, b)) = \frac{b}{r+b}$ (all other transition probabilities are zero). It can be seen as an urn with one black and one red ball in the beginning, and then a ball is chosen at random and itself and another ball of the same colour is put back in the urn. Define $M_n = \frac{R_n}{B_n + R_n}$ the fraction of red balls, which is a martingale. Show that it converges and determine the distribution of the limit.
Determine – using combinatorics – the probability to have k red balls after n steps; identify that distribution and let $n \to \infty$.