Technische Universität Berlin **WS 2010/2011**
Fakultät II – Institut f. Mathematik
Lecturer: Prof. Dr. J.-D. Deuschel hand out: Wed 26.01.2011
Assistant: Simon Wasserroth due: Wed 02.02.2011

Assignment 13
„Wahrscheinlichkeitstheorie 2 - Stochastic processes 1“

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total points: 20 Points

Problem 1 **5 Points**

(i) Show that the limit of weak convergence of probability measures is unique.

(ii) Let $\mu, \mu_1, \mu_2, \ldots$ be probability measures on $(\mathbb{R}, \mathcal{B})$. Prove or find a counterexample: μ_n converges weakly to μ if for all compact $F \subseteq \mathbb{R}$ $\limsup_{n\to\infty} \mu_n(F) \leq \mu(F)$.

Problem 2 **5 Points**

Decide which of the given sequences (μ_n) of probability measures on $(\mathbb{R}, \mathcal{B})$ converges weakly to a probability measure and determine its limit (if exists).

(i) $\mu_n = \frac{1}{n}\sum_{i=1}^n \delta_{\frac{i}{n}}$, with $\delta_.$ being the Dirac distribution.

(ii) $\mu_n = \mathrm{Bin}(n, \frac{\lambda}{n})$, for some $\lambda > 0$. $\mathrm{Bin}(n, p)$ stands for the binomial distribution.

(iii) $\mu_n = \mathcal{U}([-n, n])$, where $\mathcal{U}([a, b])$ stands for the uniform distribution.

Problem 3 **5 Points**

Let μ_n, ν_n be sequences of probability measures on $(\mathbb{R}, \mathcal{B})$. Assume for all n that $\nu_n \ll \mu_n$ with density f_n. Define on the product space $(\Omega, \mathcal{F}) = (\mathbb{R}^{\mathbb{N}}, \mathcal{B}^{\otimes\mathbb{N}})$ the measures $P = \otimes_{n\in\mathbb{N}}\mu_n$ and $Q = \otimes_{n\in\mathbb{N}}\nu_n$. Define $a_n = \int_{\mathbb{R}} \sqrt{f_n}\mathrm{d}\mu_n$.
Show that if $\prod_{i=1}^\infty a_i > 0$ then $Q \ll P$ and that $\frac{\mathrm{d}Q}{\mathrm{d}P}(\omega) = \lim_{n\to\infty}\prod_{i=1}^n f_i(\omega_i)$ P-a.s. for $\omega \in \Omega$. Hint: Define $\xi_n = f_n \circ \pi_n$ where $\pi_n : \Omega \to \mathbb{R}$ is the projection on the nth coordinate.

Next, deduce a condition for the following example: Let $0 < p, q < 1$ and $\mu_n \sim \mathrm{Ber}(p)$ and $\nu_n \sim \mathrm{Ber}(q)$. Under which condition on p and q is $Q \ll P$? **Extra:** Let $p_n \in (0, 1)$ and $\mu_n \sim \mathrm{Ber}(p_n)$, ν_n as before. Under which condition is $Q \ll P$?

Problem 4 **5 Points**

Let (X_i) be a sequence of independent random variables (on $\mathbb{R}$) with finite second moment and $\sum_{n=1}^\infty \frac{1}{n^2}\mathrm{var}(X_n) < \infty$. Show (using the martingale convergence theorem) a strong law of large numbers for the X_i. Thus, show that $\frac{1}{n}\sum_{i=1}^n (X_i - \mathbb{E}[X_i]) \to 0$ a.s.