Technische Universität Berlin **WS 2010/2011**
Fakultät II – Institut f. Mathematik
Lecturer: Prof. Dr. J.-D. Deuschel hand out: Wed 27.10.2010
Assistant: Simon Wasserroth due: Wed 03.11.2010

Assignment 2
„Wahrscheinlichkeitstheorie 2 - Stochastic processes 1“

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total points: 20 Points

Problem 1 **5 Points**
Show that the integral for elementary functions is well defined. Take some elementary function $f : \Omega \to \mathbb{R}$ on the measure space $(\Omega, \mathcal{F}, \mu)$ with $f = \sum_{i=1}^{n} \alpha_i \mathbf{1}_{A_i} = \sum_{i=1}^{m} \beta_i \mathbf{1}_{B_i}$ for some real numbers α_i, β_i and measurable sets A_i, B_i and show that its integral does not depend on the representation.

Problem 2 **5 Points**
Prove the following for functions $f_n : \Omega \to \mathbb{R}$ on some measurable space $(\Omega, \mathcal{F})$ and $\mathbb{R}$ equipped with the Borel σ-algebra.

(i) A set $A \subseteq \Omega$ is measurable iff the indicator function of the set is measurable.

(ii) Let f_1 be some measurable function. Recall the definition of atoms from the first assignment. Prove that f_1 is constant on each atom of $\mathcal{F}$.

(iii) Let f_1 and f_2 be measurable functions. Show $f_1 + f_2$ is measurable.

(iv) Let all f_n be measurable. Show $\inf_{n\in\mathbb{N}} f_n$ and $\sup_{n\in\mathbb{N}} f_n$ are measurable.

Problem 3 **5 Points**
For some measure space $(\Omega, \mathcal{F}, \mu)$ let $f_n : \Omega \to \mathbb{R}$ be a sequence of functions. Show that Fatou's Lemma remains true if we take $f_n \leq g$ with g a non negative integrable function, lim sup instead of lim inf and $\geq$ instead of $\leq$. Thus it reads then: Let f_n be a sequence of measurable functions with $f_n \leq g$ for some integrable function g. Then $\int \limsup_{n\to\infty} f_n \mathrm{d}\mu \geq \limsup_{n\to\infty} \int f_n \mathrm{d}\mu$.
Find examples that it is not true if only one of the three statements is changed.

Problem 4 **5 Points**
Let $f : [0,1]^2 \to \mathbb{R}$ with $f(x,y) = \frac{1}{\sqrt{xy}}$. f takes the value infinity on both axis. Show that, still, Fubini's theorem holds, the order of integration does not matter. Why?
Now take $g : \mathbb{R}^2 \to \mathbb{R}$ with $g(x,y) = \frac{x^2-y^2}{(x^2+y^2)^2}$ and check that Fubini's theorem does not apply and $\int_0^1 \int_0^1 g(x,y)\mathrm{d}x\mathrm{d}y \neq \int_0^1 \int_0^1 g(x,y)\mathrm{d}y\mathrm{d}x$. Why?

Reminder: Hand in your solution in groups of two students.