Technische Universität Berlin **WS 2010/2011**
Fakultät II – Institut f. Mathematik
Lecturer: Prof. Dr. J.-D. Deuschel hand out: Wed 24.11.2010
Assistant: Simon Wasserroth due: Wed 01.12.2010

Assignment 6
„Wahrscheinlichkeitstheorie 2 - Stochastic processes 1“

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total points: 20 Points

Problem 1 **5 Points**

Let $X = (X_1, \dots, X_n)$ be a random variable in $\mathbb{R}^n$. Show that the components $X_1, \dots, X_n$ are independent if and only if for the characteristic function holds $\mathbb{E}\left[e^{i(X,t)}\right] = \prod_{j=1}^n \mathbb{E}\left[e^{iX_j t_j}\right]$ for $t \in \mathbb{R}^n$. $(\cdot, \cdot)$ denotes the Euclidean scalar product.

Problem 2 **4 Points**

Let $\bar{\mathcal{B}}^{\otimes\mathbb{R}}$ be the sigma algebra on $\bar{\mathbb{R}}^{\mathbb{R}}$ which is generated by the projections $\pi_t : \bar{\mathbb{R}}^{\mathbb{R}} \to \bar{\mathbb{R}}, \pi_t(f) = f(t)$ such that $\bar{\mathcal{B}}^{\otimes\mathbb{R}} = \sigma(\pi_t^{-1}(\bar{\mathcal{B}}), t \in \mathbb{R})$. Show that $F(f) = \sup_{x\in\mathbb{R}} f(x)$ is not $\bar{\mathcal{B}}^{\otimes\mathbb{R}} - \bar{\mathcal{B}}$ measurable.

Problem 3 **7 Points**

Let $N \in \mathbb{N}$ with $N > 2$. Define for $a, b, c \in I := \{1, \dots, N\}$

$$p(a,b,c) := \begin{cases} 1 & : \quad a = b = c \\ \frac{1}{N-2} & : \quad a,\ b,\ c \text{ are pairwise different} \\ 0 & : \quad \text{else} \end{cases}$$

Define for $n \geq 2$ and $i_j \in I$ the map

$$K_n : ((i_1, \dots, i_n), i_{n+1}) \mapsto p(i_{n-1}, i_n, i_{n+1})$$

a) Show that K_n can be seen as a Markov kernel. Write down the corresponding measure spaces.

b) Let Q_1 be the uniform distribution on I, $Q_2 := Q_1 \otimes Q_1$, and $Q_{n+1} := Q_n \otimes K_n$ for $n \geq 2$. By the theorem of Ionescu-Tulcea there is a stochastic process $(X_n)_{n\in\mathbb{N}}$ with finite dimensional distributions Q_n. Show

(i) $(X_n)_{n\in\mathbb{N}}$ is not a Markov chain.

(ii) $(X_n, X_{n+1})_{n\in\mathbb{N}}$ is a Markov chain on I^2.

Problem 4 **4 Points**

Find out and prove if the following sets are in $\mathcal{B}(\mathbb{R})^{\otimes \mathbb{N}}$.

(i) The set of all sequences converging to zero.

(ii) The set of all converging sequences.

Reminder: Hand in your solution in groups of two students.