Technische Universität Berlin
Fakultät II – Institut f. Mathematik
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WS 2010/2011
hand out: Wed 01.12.2010
due: Wed 08.12.2010

Assignment 7
„Wahrscheinlichkeitstheorie 2 - Stochastic processes 1“

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total points: 20 Points

Problem 1 — 5 Points

Let $X_1, X_2, \dots, X_n$ be iid random variables with finite expectation. Show for all $n \in \mathbb{N}$

$$\mathbb{E}\left[X_1 \,|\sigma(S_n)\right] = \frac{S_n}{n} \;\; \mathbb{P}-\text{almost surely}$$

Define $S_n := \sum_{i=1}^n X_i$.

Problem 2 — 5 Points

Let $\Omega = \{a, b, c\}$. Find a random variable X on Ω and sigma algebras $\mathcal{F}_1$ and $\mathcal{F}_2$ such that

$$\mathbb{E}\left[\mathbb{E}\left[X \,|\mathcal{F}_2\right] |\mathcal{F}_1\right] \neq \mathbb{E}\left[\mathbb{E}\left[X \,|\mathcal{F}_1\right] |\mathcal{F}_2\right]$$

on a set of positive probability.

Problem 3 — 5 Points

Let X_1, X_2 be independent and both $\exp(\lambda)$-distributed. Show that

$$\mathbb{E}\left[X_1 \wedge X_2 \,|X_1\right] = \lambda^{-1}(1 - e^{-\lambda X_1}).$$

Problem 4 — 5 Points

Let $Q_1 = \mathcal{N}(\mu, \Sigma^2)$ be the normal distribution $\mathcal{N}$ with parameters (μ, Σ^2), where $\Sigma^2 \in \mathbb{R}^{d\times d}$ is a symmetric positive definit matrix. Define the Markov kernel $K : \mathbb{R}^d \times \mathcal{B}^{\otimes d} \to [0, 1]$ by $K(x, \cdot) = \mathcal{N}(Ax, \tilde{\Sigma}^2)$, where A is some $d \times d$ matrix and $\tilde{\Sigma}^2$ is a symmetric positive definit matrix. Now let $Q_2 = Q_1 \otimes K$. Calculate the distribution Q_2 explicitly.
Extra: Determine the distribution of $Q_2\pi_2^{-1} = Q_1K$.

Reminder: Hand in your solution in groups of two students.