Technische Universität Berlin **WS 2012/2013**
Fakultät II – Institut für Mathematik
Lecturer: Prof. Dr. W. König hand out: Wed, October 17th, 2012
Assistant: M. Wilke Berenguer due: Wed, Oct 24th, 2012

Assignment 1
„Wahrscheinlichkeitstheorie 2 - Stochastic processes 1“
measure spaces and measurable functions

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total points: 20 Points

All the exercises must be treated in groups of two to three students, where three is preferred.

Problem 1 *(Combinations of measures)* **6 Points**

(i) Prove that any countable linear combination of measures with nonnegative coefficients is a measure.

(ii) Prove that the set of probability measures on a given measurable space is convex.

(iii) Let Ω be a countable set, $\mathfrak{P}$ the set of probability measures on $(\Omega, \mathcal{P}(\Omega))$. Find all extremal points of $\mathfrak{P}$ and prove that any $\mu \in \mathfrak{P}$ can be written as a countable linear combination of extremal points.

Remark: $\mu \in \mathfrak{P}$ is called *extremal* if from the existence of $\mu_1, \mu_2 \in \mathfrak{P}$ and $\alpha \in [0,1]$ such that $\mu = \alpha\mu_1 + (1-\alpha)\mu_2$ it follows that $\mu = \mu_1$ or $\mu = \mu_2$.

Problem 2 *(Weak Dynkin systems)* **5 Points**

A family $\mathfrak{D}$ of subsets of a non-empty set Ω ist called a *weak Dynkin system* if the following hold true:

(i) $\Omega \in \mathfrak{D}$

(ii) $D \in \mathfrak{D} \;\Rightarrow D^c \in \mathfrak{D}$

(iii) For any increasing sequence $(D_n)_{n\in\mathbb{N}}$ of elements of $\mathfrak{D}$ (i.e., $D_n \subseteq D_{n+1}$ for any $n \in \mathbb{N}$), we have $\bigcup_{n\in\mathbb{N}} D_n \in \mathfrak{D}$.

Analyse whether or not the assertions of Lemma 6.2.2 and Theorem 6.2.3 also hold for weak Dynkin systems, and prove your respective claims.

Remark: For an English version of these statements and proofs, see Theorems 1.18 and 1.19 in "Probability Theory: A Comprehensive Course" by Achim Klenke (Springer).

Problem 3 *(Completion of measurable spaces)* **5 Points**

Consider a measure space $(\Omega, \mathcal{F}, \mu)$ and recall the definition of the completion of a measure space in Remark 6.2.12 from the lecture (recall that $\mathcal{G}$ was called $\overline{\mathcal{F}^\mu}$ in the lecture). Prove all the necessary steps to show that $(\Omega, \overline{\mathcal{F}^\mu}, \overline{\mu})$ is indeed a well-defined, complete measure space that extends $(\Omega, \mathcal{F}, \mu)$.

Problem 4 *(Combinations of measurable functions)* **4 Points**

Let $(\Omega, \mathcal{F})$ be a measurable space. Prove the following assertions:

(i) If f and g are measurable real functions such that for all $\omega \in \Omega$ we have $g(\omega) \neq 0$, then f/g is a measurable real function, too.

(ii) Let $(f_n)_{n\in\mathbb{N}}$ be a sequence of measurable numerical functions. Then $\sup_{n\in\mathbb{N}} f_n$ is also a measurable numerical function.