Technische Universität Berlin **WS 2012/2013**
Fakultät II – Institut für Mathematik
Lecturer: Prof. Dr. W. König
Assistant: M. Wilke Berenguer

hand out: Wed Oct 24th 2012
due: Wed, Oct 31st, 2012

Assignment 2
„Wahrscheinlichkeitstheorie 2 - Stochastic processes 1“
integration, image measures and densities

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total points: 20 Points

Problem 1 *(Equality μ-almost everywhere)* **2 Points**
Let $(x_n)_{n\in\mathbb{N}}$ be a sequence of real numbers. Then $\mu := \sum_{n\in\mathbb{N}} \delta_{x_n}$ is a measure on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$. Prove that two functions $f, g\colon \mathbb{R} \to \mathbb{R}$ are μ-almost-everywhere equal if and only if $f(x_n) = g(x_n)$ for all $n \in \mathbb{N}$.

Problem 2 *(Measurability and integration)* **5 Points**
Let Ω be an uncountable set and $\mathcal{S} := \{A \subseteq \Omega \mid A \text{ is countable or } A^{\mathrm{c}} \text{ is countable}\}$.

(i) Prove that $\mathcal{S}$ is a σ-algebra.

(ii) Prove that a numerical function $f\colon \Omega \to \overline{\mathbb{R}}$ is measurable if and only if it is constant on a set $A_f \in \mathcal{S}$ whose complement is countable.

(iii) Let $\mu\colon \mathcal{S} \to [0, \infty]$ be defined as

$$\mu(A) = \begin{cases} 0 & \text{if } A \text{ is countable,} \\ \infty & \text{otherwise.} \end{cases}$$

Show that μ is a measure on $(\Omega, \mathcal{S})$.

(iv) Determine all μ-integrable functions and their integrals.

Problem 3 *(Integration w.r.t. a measure with a density)* **4 Points**
Prove the following statement by measure-theoretic induction:
Let μ and ν be two measures on a measurable space $(\Omega, \mathcal{F})$. Assume that the function $\varphi\colon \Omega \to [0, \infty[$ is a density of ν with respect to μ. Then we have

$$\int g \, \mathrm{d}\nu = \int g\varphi \, \mathrm{d}\mu$$

for all measurable functions g on $(\Omega, \mathcal{F})$ in the following sense: If one of the two expressions exists, then so does the other and they are equal.

Problem 4 *(Measurability)* **4 Points**

Let $(\Omega, \mathcal{F})$ be a measurable space and $f\colon \Omega \to [0, \infty[$ a non-negative function. Define $A := \{(\omega, y) \in \Omega \times \mathbb{R} \mid 0 < y < f(\omega)\}$ and prove that:

$$f \text{ is } \mathcal{F} - \mathcal{B}(\mathbb{R})\text{-measurable} \qquad \Longleftrightarrow \qquad A \in \sigma(\mathcal{F} \times \mathcal{B}(\mathbb{R})),$$

where we wrote $\mathcal{F} \times \mathcal{B}(\mathbb{R})$ for the set of all sets $A \times B$ with $A \in \mathcal{F}$ and $B \in \mathcal{B}(\mathbb{R})$.

For the following exercise you will need the "Transformationssatz" for Lebesgue-integrable functions:

Theorem. *Let U and V be two open subsets of $\mathbb{R}^n$ and $\varphi\colon U \to V$ a $\mathcal{C}^1$-invertible map. Then a function $f\colon V \to \overline{\mathbb{R}}$ is Lebesgue-integrable if and only if the map $f \circ \varphi |\det(D\varphi)|$ is Lebesgue-integrable over U. In this case,*

$$\int_U f(\varphi(x)) |\det(D\varphi)| \, \lambda(dx) = \int_V f(y) \, \lambda(dy).$$

Problem 5 *("Transformationssatz" for Lebesgue-integrable functions)* **5 Points**

We use the notation from the theorem above.

(i) Prove that the map $\mu\colon \mathcal{B}(\mathbb{R}) \to \mathbb{R}$ defined by $\mu(A) = \lambda(\varphi(A))$ is a measure and decide whether it has a density with respect to the Lebesgue-measure. Give a density in case of existence.

(ii) Use the theorem above or your result from (i) to prove that

$$\varphi_{b,\Sigma}(x) := \frac{1}{\sqrt{(2\pi)^n \det \Sigma}} \exp\left(-\frac{1}{2}(x-b)^{\mathrm{T}} \Sigma^{-1} (x-b)\right), \qquad x \in \mathbb{R}^n,$$

is indeed a density of the image-measure $\mathcal{N} \circ f^{-1}$ of the n-dimensional standard normal distribution $\mathcal{N}$ under the affine map $f\colon \mathbb{R}^n \to \mathbb{R}^n$, $x \mapsto Ax + b$ for an invertible matrix $A \in \mathbb{R}^{n \times n}$ and a vector $b \in \mathbb{R}^n$, where $\Sigma = AA^{\mathrm{T}}$.