Technische Universität Berlin **WS 2012/2013**
Fakultät II – Institut für Mathematik
Lecturer: Prof. Dr. W. König hand out: Wed, Nov 14th 2012
Assistant: M. Wilke Berenguer due: Wed, Nov 21st, 2012

Assignment 5
„Wahrscheinlichkeitstheorie 2 - Stochastic processes 1“
independence and Kolmogorov's 0-1-law

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total points: 20 Points

Problem 1 *(Limit superior and inferior of set sequences)* **4 Points**
Let $(\Omega, \mathcal{F}, \mu)$ be a measure space and $(A_n)_{n\in\mathbb{N}}$ a sequence in $\mathcal{F}$. We define

$$\liminf_{n\to\infty} A_n = \bigcup_{n\in\mathbb{N}} \bigcap_{m=n}^{\infty} A_m \qquad \text{and} \qquad \limsup_{n\to\infty} A_n = \bigcap_{n\in\mathbb{N}} \bigcup_{m=n}^{\infty} A_m.$$

(i) Prove that $\mu(\liminf_{n\to\infty} A_n) \le \liminf_{n\to\infty} \mu(A_n)$.

(ii) If μ is finite, then prove that $\mu(\limsup_{n\to\infty} A_n) \ge \limsup_{n\to\infty} \mu(A_n)$.

Problem 2 *(Functions of independent random variables)* **6 Points**
Prove or disprove the following three assertions for any two random variables X and Y:

(i) If X and Y are independent, then so are $f(X)$ and $g(Y)$ for any measurable functions $f, g\colon \mathbb{R} \to \mathbb{R}$.

(ii) If X and Y are uncorrelated, then so are $f(X)$ and $g(Y)$ for any measurable functions $f, g\colon \mathbb{R} \to \mathbb{R}$.

(iii) X, Y are independent if and only if $\mathbb{E}[f(X)g(Y)] = \mathbb{E}[f(X)]\mathbb{E}[g(Y)]$ for any measurable functions $f, g\colon \mathbb{R} \to \mathbb{R}$.

Problem 3 *(Percolation)* **4 Points**
Consider the graph with vertex set $\mathbb{Z}^d$ and edges between any two nearest neighbours, i.e., between any $x, y \in \mathbb{Z}^d$ such that $\|x - y\|_1 = 1$. Fix $p \in [0,1]$. Any edge is 'open' with a probability p and 'closed' otherwise, and the opennesses of the edges are all independent. If you consider this lattice as a set of pipes, water can only flow through the pipes (or edges) that are 'open'.
For $d = 1$, prove that the event that there exists a point from which water can flow to infinity (i.e., to either $+\infty$ or to $-\infty$) has probability 0 or 1.

Remark: If you prove the assertion for general $d \ge 2$, you receive **additional** 2 points.

Problem 4 *(Cesàro limits)* **6 Points**

Let $(X_n)_{n\in\mathbb{N}}$ be a sequence of independent random variables. Prove that the Cesàro limits

$$\underline{Y} := \liminf_{n\to\infty} \frac{1}{n}\sum_{k=1}^{n} X_k \qquad \text{and} \qquad \overline{Y} = \limsup_{n\to\infty} \frac{1}{n}\sum_{k=1}^{n} X_k$$

are almost surely constant taking values in $\mathbb{R}\cup\{-\infty,\infty\}$.