Technische Universität Berlin **WS 2012/2013**
Fakultät II – Institut für Mathematik
Lecturer: Prof. Dr. W. König hand out: Wed, Dec 5th 2012
Assistant: M. Wilke Berenguer due: Wed, Dec 12th, 2012

Assignment 8
„Wahrscheinlichkeitstheorie 2 - Stochastic processes 1“
Martingale Theory

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total points: 20 Points

For this assignment to be marked you need to give the information asked for in the table below. This needs to be done by **any current** group member. Only those people who have completed this information will continue on the list of students of this class. If the cardinality of your group has decreased and you are interested in merging with another group, please also give your email for me to contact you.
For the column "examination month" please choose "February", "March", "April", or "later", depending on the time-period you prefer for your final exam. (The months refer to the year 2013.)

Surname	Name	degree program	Matrikelnummer	examination month

Problem 1 *(More properties of pre-τ sigma-fields)* **5 Points**
Let τ and σ be stopping times with respect to the filtration $\mathbb{F}$. Prove the following statements:

(i) $\mathcal{F}_{\tau\wedge\sigma} = \mathcal{F}_\tau \cap \mathcal{F}_\sigma$.

(ii) For any $\mathbb{F}$-adapted process $(X_n)_{n\in\mathbb{N}}$, if $\tau(\omega) < \infty$ for all $\omega \in \Omega$, then X_τ is $\mathcal{F}_\tau$-measurable.

(iii) For $n \in \mathbb{N}$, define $\mathcal{F}_n^\tau := \mathcal{F}_{n\wedge\tau}$ and $\mathcal{F}_\infty^\tau = \sigma(\mathcal{F}_n^\tau \colon n \in \mathbb{N})$. Then $\mathcal{F}_\infty^\tau = \mathcal{F}_\tau$.

Problem 2 *(Absorption probabilities for simple random walk)* **5 Points**
Let $(S_n)_{n\in\mathbb{N}_0}$ be the simple symmetric random walk on $\mathbb{Z}$ (i.e., jumping independently with equal probability to any of the two nearest neighbours at any time), starting in 0. For $x \in \mathbb{Z}$, let $\tau_x := \inf\{n \in \mathbb{N} \colon S_n = x\}$ be the hitting time of x. Because of recurrence (see Theorem 9.3.9), we know that τ_x is finite almost surely.
Now fix $a, b \in \mathbb{N}$ and put $\tau = \tau_{-a} \wedge \tau_b$. Prove that $\mathbb{P}(\tau = \tau_{-a}) = \frac{b}{a+b}$.

Hint: Approximate τ with $\tau \wedge n$ for $n \to \infty$.

Problem 3 Doob-Meyer-decomposition **5 Points**

Let $(X_n)_{n\in\mathbb{N}}$ be an $\mathbb{F}$-Submartingale. Prove that then there exist an $\mathbb{F}$-previsible and increasing process $(A_n)_{n\in\mathbb{N}}$ and an $\mathbb{F}$-martingale $(M_n)_{n\in\mathbb{N}}$, such that $M_1 = 0, A_1 = 0$ almost surely and

$$\forall n \in \mathbb{N}: \qquad X_n = M_n + A_n + X_0 \qquad \text{almost surely.}$$

Problem 4 *(A directed polymer in a random environment)* **5 Points**

Let $\eta = (\eta(n,x))_{n\in\mathbb{N}_0, x\in\mathbb{Z}^d}$ be a family of i.i.d. random variables such that $E[\exp(\lambda\eta(n,x))] < \infty$ for any $\lambda > 0$, where E denotes expectation with respect to the random environment η. Furthermore, let $(S_n)_{n\in\mathbb{N}_0}$ be a simple random walk in $\mathbb{Z}^d$ starting in 0. The random walk and the environment are assumed independent. Define $Y_n := \sum_{k=0}^{n} \eta(k, S_k)$. For a fixed $\beta > 0$, we put $Z_n = \mathbb{E}[\exp(\beta Y_n)]$, where $\mathbb{E}$ is the expectation with respect to the random walk. Consider the filtration $\mathbb{F} = (\mathcal{F}_n)_{n\in\mathbb{N}}$ given by $\mathcal{F}_n := \sigma(\eta(k,x) \mid k \le n, x \in \mathbb{Z}^d)$.

Find a sequence $(C_n)_{n\in\mathbb{N}_0}$ of positive numbers such that $(Z_n/C_n)_{n\in\mathbb{N}_0}$ is an $(\mathcal{F}_n)_{n\in\mathbb{N}_0}$-martingale.

Hint: Compare to an exponential martingale.