Technische Universität Berlin **WS 2012/2013**

Fakultät II – Institut für Mathematik

Lecturer: Prof. Dr. W. König hand out: Wed, Dec 12th 2012

Assistant: M. Wilke Berenguer due: Wed, Dec 19th, 2012

Assignment 9
„Wahrscheinlichkeitstheorie 2 - Stochastic processes 1“
Martingale Limit Theory

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total points: 20 Points

Problem 1 *(Criteria for uniform integrability)* **5 Points**

Let $\{X_t \colon t \in I\}$ be a family of real random variables. Prove the following.

(i) $\{X_t \colon t \in I\}$ is uniformly integrable if and only if $\sup_{t\in I} \mathbb{E}[|X_t|] < \infty$ and for any $\varepsilon > 0$ there is a $\delta > 0$ such that $\mathbb{E}[|X_t|\mathbb{1}_A] < \varepsilon$ for any $t \in I$ and any $A \in \mathcal{F}$ satisfying $\mathbb{P}(A) < \delta$.

(ii) Suppose there is a measurable function $G \colon [0,\infty[\to [0,\infty[$ satisfying $\lim_{x\to\infty} \frac{G(x)}{x} = \infty$ and $\sup_{t\in I} \mathbb{E}[G(|X_t|)] < \infty$. Then $\{X_t \colon t \in I\}$ is uniformly integrable.

Remark: In (ii), also the reversed implication holds.

Problem 2 *(Martingale proof of Kolmogorov's 0-1-law)* **5 Points**

Prove Kolmogorov's 0-1-law (Theorem 7.2.8) using martingale methods. That is, let $(\mathcal{F}_n)_{n\in\mathbb{N}}$ be a sequence of independent σ-fields and $\mathcal{T}_\infty$ the corresponding tail σ-field, then prove that $\mathbb{P}(A) \in \{0,1\}$ for any $A \in \mathcal{T}_\infty$, using a suitable martingale. You are allowed to use the independence of $\mathcal{T}_\infty$ and $\mathcal{F}_1 \vee \cdots \vee \mathcal{F}_n$ for all $n \in \mathbb{N}$.

Problem 3 **5 Points**

Let $(X_n)_{n\in\mathbb{N}}$ be a non-negative supermartingale with respect to the filtration $\mathbb{F} = (\mathcal{F}_n)_{n\in\mathbb{N}}$. According to Corollary 1.4.4, there exists an integrable random variable X_∞ such that $\lim_{n\to\infty} X_n = X_\infty$ almost surely. Prove that $X_n \geq \mathbb{E}[X_\infty \mid \mathcal{F}_n]$ a. s. for any $n \in \mathbb{N}$.

Hint: If you want to use Fatou's lemma for conditional expectations, you must prove it.

Problem 4 *(A strong law of large numbers)* **5 Points**

Let $(X_n)_{n\in\mathbb{N}}$ be a sequence of independent real random variables with finite second moments and $(a_n)_{n\in\mathbb{N}}$ an increasing unbounded sequence of positive numbers.
Using martingale methods, prove that $\sum_{i=1}^\infty \mathbb{V}(X_i)/a_i^2 < \infty$ implies

$$\lim_{n\to\infty} \frac{1}{a_n} \sum_{i=1}^n (X_i - \mathbb{E}[X_i]) = 0 \text{ a.s.}$$

Hint: You may use *Kronecker's lemma* stating that for a sequence $(a_n)_{n\in\mathbb{N}}$ as above and any real sequence $(b_n)_{n\in\mathbb{N}}$, we have that if $\sum_{i=1}^\infty b_i/a_i$ exists and is finite, then $\lim_{n\to\infty} a_n^{-1} \sum_{i=1}^n b_i = 0$.