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Approximation Algorithms (ADM III)
Winter term 2010

Problem Set 4

(due date: *December 8, 2010*)

Exercise 4.1 **4 points**

In the *maximum directed cut problem* (sometimes called MAX DICUT) we are given as input a directed graph $G = (V, A)$. Each directed arc $(i, j) \in A$ has nonnegative weight $w_{ij} \geq 0$. The goal is to partition V into two sets U and $W = V - U$ so as to maximize the total weight of the arcs going from U to W (that is, arcs (i, j) with $i \in U$ and $j \in W$). Give a randomized $\frac{1}{4}$-approximation algorithm for this problem.

Exercise 4.2 **4 points**

Consider the non-linear randomized rounding algorithm for MAX SAT as discussed in class. Prove that using randomized rounding with the linear function $f(y_i) = \frac{1}{2}y_i + \frac{1}{4}$ also gives a $\frac{3}{4}$-approximation algorithm for MAX SAT.

Exercise 4.3 **6 points**

Consider again the maximum directed cut problem from the exercise above.

(a) Show that the following integer program models the maximum directed cut problem:

$$\begin{aligned} \text{maximize} \quad & \sum_{(i,j)\in A} w_{ij} z_{ij} & \\ \text{subject to} \quad & z_{ij} \leq x_i, & \forall (i,j) \in A, \\ & z_{ij} \leq 1 - x_j, & \forall (i,j) \in A, \\ & x_i \in \{0,1\}, & \forall i \in V, \\ & 0 \leq z_{ij} \leq 1, & \forall (i,j) \in A. \end{aligned}$$

(b) Consider a randomized rounding algorithm for the maximum directed cut problem that solves a linear programming relaxation of the integer program and puts vertex $i \in U$ with probability $1/4 + x_i/2$. Show that this gives a randomized 1/2-approximation algorithm for the maximum directed cut problem.

Exercise 4.4 **6 points**

This exercise introduces a deterministic rounding technique called *pipage rounding*. To illustrate this technique, we will consider the problem of finding a maximum cut in a graph with a constraint on the size of each part. In the maximum cut problem, we are given as input an undirected graph $G = (V, E)$ with nonnegative weights $w_{ij} \geq 0$ for all $(i, j) \in E$. We wish to partition the vertex set into two parts U and $W = V - U$ so as to maximize the weight of the edges whose two endpoints are in different parts. We will also assume that we are given an integer $k \leq |V|/2$, and we must find a partition such that $|U| = k$.

(a) Show that the following *nonlinear* integer program models the maximum cut problem with a constraint on the size of the parts:

$$
\begin{aligned}
\text{maximize} \quad & \sum_{(i,j)\in E} w_{ij}(x_i + x_j - 2x_i x_j) \\
\text{subject to} \quad & \sum_{i \in V} x_i = k, \\
& x_i \in \{0, 1\}, \qquad \forall i \in V.
\end{aligned}
$$

(b) Show that the following linear program is a relaxation of the problem:

$$
\begin{aligned}
\text{maximize} \quad & \sum_{(i,j)\in E} w_{ij} z_{ij} \\
\text{subject to} \quad & z_{ij} \leq x_i + x_j, && \forall (i,j) \in E, \\
& z_{ij} \leq 2 - x_i - x_j, && \forall (i,j) \in E, \\
& \sum_{i \in V} x_i = k, \\
& 0 \leq z_{ij} \leq 1, && \forall (i,j) \in E, \\
& 0 \leq x_i \leq 1, && \forall i \in V.
\end{aligned}
$$

(c) Let $F(x) = \sum_{(i,j)\in E} w_{ij}(x_i + x_j - 2x_i x_j)$ be the objective function from the nonlinear integer program. Show that for any (x, z) that is a feasible solution to the linear programming relaxation, $F(x) \geq \frac{1}{2} \sum_{(i,j)\in E} w_{ij} z_{ij}$.

(d) Argue that given a fractional solution x, for two fractional variables x_i and x_j, it is possible to increase one by $\epsilon > 0$ and decrease the other by ϵ such that $F(x)$ does not decrease and one of the two variables becomes integer.

(e) Use the arguments above to devise a $\frac{1}{2}$-approximation algorithm for the maximum cut problem with a constraint on the size of the parts.