TU Berlin
Institut für Mathematik
Prof. David Williamson
Jannik Matuschke

Problem Set 6

(due date: *January 26, 2011*)

Exercise 6.1 **5 points**

In the *Steiner k-cut problem*, we are given an undirected graph $G = (V, E)$, costs $c_e \geq 0$ for all $e \in E$, a set of terminals $T \subseteq V$, and a positive integer $k \leq |T|$. The goal of the problem is to partition the vertices into k sets $S_1, \ldots, S_k$ such that each set contains at least one terminal (that is, $S_i \cap T \neq \emptyset$ for $i = 1, \ldots, k$) and to minimize the weight of the edges with endpoints in different parts. Given a partition $\mathcal{P} = \{S_1, \ldots, S_k\}$, let $c(\mathcal{P})$ be the total cost of the edges that have endpoints in different parts of the partition.

Consider the following greedy algorithm for the Steiner k-cut problem: We start with $\mathcal{P} = \{V\}$. As long as $\mathcal{P}$ does not have k parts, we consider each set $S \in \mathcal{P}$ with $|S \cap T| \geq 2$, consider each pair of terminals in $S \cap T$, and compute the minimum-cost cut between that pair of terminals. We then choose the minimum-cost cut found overall by this procedure; note that this breaks some set $S \in \mathcal{P}$ into two parts. We replace S in $\mathcal{P}$ with these two new parts, and continue.

(a) Let $\mathcal{P}_i$ be the contents of the partition found by the algorithm when it has i parts. Let $\hat{\mathcal{P}} = \{V_1, V_2, \ldots, V_i\}$ be any valid partition into i parts (that is, $V_j \cap T \neq \emptyset$ for $j = 1, \ldots, i$). Show that

$$c(\mathcal{P}_i) \leq \sum_{j=1}^{i-1} \sum_{e \in \delta(V_j)} c_e.$$

(b) Use the above to show that this greedy algorithm is a $\left(2 - \frac{2}{k}\right)$-approximation algorithm for the Steiner k-cut problem.

Exercise 6.2 **5 points**

In the *universal traveling salesman problem*, we are given as input a metric space (V, d) and must construct a tour π of the vertices. Let π_S be the tour of the vertices $S \subseteq V$ given by visiting them in the order given by the tour π. Let OPT_S be the value of an optimal tour on the metric space induced by the vertices $S \subseteq V$. The goal of the problem is to find a tour π that minimizes π_S / OPT_S over all $S \subseteq V$; in other words, we'd like to find a tour such that for any subset $S \subseteq V$, visiting the vertices of S in the order given by the tour is close in value to the optimal tour of S.

Show that if (V, d) is a tree metric, then it is possible to find a tour π such that $\pi_S = \mathrm{OPT}_S$ for all $S \subseteq V$.

Exercise 6.3 **5 points**

In the *capacitated dial-a-ride problem*, we are given a metric (V, d), a vehicle of capacity C, a starting point $r \in V$, and k source-sink pairs s_i-t_i for $i = 1, \ldots, k$, where $s_i, t_i \in V$. At each source s_i there is an item that must be delivered to the sink t_i by the vehicle. The vehicle can carry at most C items at a time. The goal is to find the shortest possible tour for the vehicle that starts at r, delivers each item from its source to its destination without exceeding the vehicle capacity, then returns to r; note that such a tour may visit a node of V multiple times. We assume that the vehicle is allowed to temporarily leave items at any node in V.

(a) Suppose that the metric (V, d) is a tree metric (V, T). Give a 2-approximation algorithm for this case. (Hint: How many times must each edge $(u, v) \in T$ be traversed going from u to v, and going from v to u? Give an algorithm that traverses each edge at most twice as many times as it needs to.)

(b) Give a randomized $O(\log |V|)$-approximation algorithm for the capacitated dial-a-ride problem in the general case.

Exercise 6.4 **5 points**

Let $C_n = (V, E)$ be a cycle on n vertices, and let d_{uv} be the distance between $u, v \in V$ on C_n. Show that for any tree metric (V, T) on the same set of vertices V, there must exist a pair of vertices $u, v \in V$ such that $d_{uv} = 1$, but $T_{uv} \geq n - 1$. To do this, suppose that of all trees T with optimal distortion, T has the minimum total length. Show that T must be a path of vertices of degree two, then conclude the statement above.