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Approximation Algorithms (ADM III)
Winter term 2010

Problem Set 7

(due date: *February 9, 2011*)

Remark: All exercises of this problem set deal with the properties of *basic feasible solutions* of certain linear programs. You can find a short definition of this term at the beginning of Chapter 11 in the book.

Exercise 7.1 **5 points**

Recall the LP relaxation we used for the survivable network design problem:

$$\begin{aligned} \text{minimize} \quad & \sum_{e \in E} c_e x_e \\ \text{subject to} \quad & \sum_{e \in \delta(S)} x_e \geq f(S), \qquad \forall S \subseteq V, \\ & x_e \geq 0 \qquad e \in E. \end{aligned}$$

We claimed, but did not prove, the following: For any basic feasible solution x to the linear program with f a weakly supermodular function, there is a collection $\mathcal{L}$ of subsets of vertices with the following properties:

(1) For all $S \in \mathcal{L}$, S is tight.

(2) The vectors $\chi_{\delta(S)}$ for $S \in \mathcal{L}$ are linearly independent.

(3) $|\mathcal{L}| = |\{e \in E : x_e > 0\}|$.

(4) The collection $\mathcal{L}$ is laminar.

First, prove the following. Given two tight sets A and B, one of the following two statements must hold:

- $A \cup B$ and $A \cap B$ are tight, and $\chi_{\delta(A)} + \chi_{\delta(B)} = \chi_{\delta(A \cap B)} + \chi_{\delta(A \cup B)}$; or
- $A - B$ and $B - A$ are tight, and $\chi_{\delta(A)} + \chi_{\delta(B)} = \chi_{\delta(A-B)} + \chi_{\delta(B-A)}$.

Then prove the theorem above. You may assume that any basic feasible solution fulfills the properties (1) to (3).

Exercise 7.2 **5 points**

Consider the following LP relaxation for the traveling salesman problem:

$$\begin{aligned} \text{minimize} \quad & \sum_{e \in E} c_e x_e \\ \text{subject to} \quad & \sum_{e \in \delta(S)} x_e \geq 2, && \forall S \subset V, S \neq \emptyset, \\ & 0 \leq x_e \leq 1, && \forall e \in E. \end{aligned}$$

Show that for any basic feasible solution x to the linear program, there must exist some $e \in E$ such that $x_e = 1$.

Exercise 7.3 **5 points**

The *minimum k-edge-connected subgraph problem* takes as input an undirected graph $G = (V, E)$ and a positive integer k. The goal is to find the smallest set of edges $F \subseteq E$ such that there are at least k edge-disjoint paths between each pair of vertices.

Consider the following linear programming relaxation of the problem:

$$\begin{aligned} \text{minimize} \quad & \sum_{e \in E} x_e \\ \text{subject to} \quad & \sum_{e \in \delta(S)} x_e \geq k, && \forall S \subseteq V, \\ & 0 \leq x_e \leq 1 && e \in E. \end{aligned}$$

(a) Prove that the linear program is indeed a relaxation of the problem.

(b) Prove that the linear program can be solved in polynomial time.

(c) Suppose we obtain a basic optimal solution to the LP relaxation and round up *every* fractional variable to 1. Prove that this gives a $(1 + \frac{4}{k})$-approximation algorithm for the problem.